The law of a stochastic integral with two independent fractional Brownian motions

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Abstract

Using the tools of the stochastic integration with respect to the fractional Brownian motion, we obtain the expression of the characteristic function of the random variable $\int_0^1 B_s^\alpha dB_s^H$ where B^α and B^H are two independent fractional Brownian motions with Hurst parameters $\alpha \in (0,1)$ and $H > \frac{1}{2}$ respectively. The two-parameter case is also considered.

1 Introduction

In general, it is difficult to compute the law of a stochastic integral with respect to the Wiener process when the integrand is not deterministic. There are some known results in particular cases. Let us recall the context. Consider W^1 and W^2 two independent Brownian motions. In [6] and [17] the authors studied the law of the random variable

$$\alpha \int_0^1 W_s^1 dW_s^2 + \beta \int_0^1 W_s^2 dW_s^1.$$

When $\alpha = 1$ and $\beta = 0$ they showed that the characteristic function of the stochastic integral $\int_{[0,1]} W_s^1 dW_s^2$ is given by

$$\Phi(t) = \left(\cosh^2\left(\frac{t}{2}\right) + \sinh^2\left(\frac{t}{2}\right)\right)^{-\frac{1}{2}}. \tag{1}$$

In the two-parameter case in [9] (see also [11]) the authors proved that the characteristic function of the integral $\int_{[0,1]^2} W^1_{\underline{s}} dW^2_{\underline{s}}$ (here W^1 and W^2 denotes two independent Brownian sheets) is given by

$$\Phi(t) = \prod_{k\geq 1} \cosh^{-\frac{1}{2}} \left(\frac{2t}{(2k-1)\pi} \right). \tag{2}$$

The aim of the present work is develop a similar study for the fractional Brownian motion. The recent development of the stochastic integration with respect to the fractional Brownian motion (fBm) gives the tools for this analysis. Concretely, we will consider two independent fractional Brownian motion B^H and B^α with Hurst parameter $\alpha \in (0,1)$ and $H > \frac{1}{2}$, and we will find an explicit expression for the characteristic function of the stochastic integral $\int_0^1 B^\alpha_s dB^H_s$. We mention that this kind of integrals appears in the study of stochastic wave equations with fractional noise (see [5]). Related results on the law of this integral have also been proved in [7].

2 Preliminaries: Fractional Brownian motion and Wiener integrals

Let $T = [0,1]$ the unit interval and let $\left(B^H_t\right)_{t\in T}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Denote by R^H its covariance

$$R^H(t,s) = E\left(B^H_t B^H_s\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

We denote by $\mathcal{H}(H) := \mathcal{H}$ the canonical space of the fractional Brownian motion B^H. That is, $\mathcal{H}$ is the closure of the linear span of the indicator functions $\{1_{[0,t]}, t \in T\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R^H(t,s).$$

The structure of the Hilbert space $\mathcal{H}$ varies upon the values of the Hurst parameter. Let us recall some basic facts about this space.

- If $H > \frac{1}{2}$ the elements of $\mathcal{H}$ may be not functions but distributions of negative order (see [13]). Therefore, it is of interest to know significant subspaces of functions contained in it.

  Define the function

  $$\theta^H(s,t) = H(2H-1)|s-t|^{2H-2} \tag{3}$$

  and let $L^2_H(T)$ be the set of function $f : T \to \mathbb{R}$ such that $\int_T \int_T |f(u)||f(v)|\theta(u,v)dudv < \infty$, endowed with the scalar product

  $$\langle f, g \rangle_H = \int_T \int_T f(u)g(v)\theta(u,v)dudv. \tag{4}$$

It has been proved in [13] that $L_H^2(T)$ is a strict subset of $\mathcal{H}$ and the scalar products $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ coincide on $L_H^2(T)$. Moreover, we have the following inclusion

$$L^{\frac{1}{H}}(T) \subset L_H^2(T) \subset \mathcal{H}. \tag{5}$$

- If $H < \frac{1}{2}$, then $\mathcal{H}$ is a set of functions; it coincides actually with the set $I_{T-}^{\frac{1}{2}-H}\left(L^2(T)\right)$ where $I_{T-}^{\frac{1}{2}-H}$ is the fractional integral of order $\frac{1}{2}-H$ (see [8], [1], [13]). A significant subspace of $\mathcal{H}$ is the set of Hölder continuous functions of order $\frac{1}{2}-H+\varepsilon$ for all $\varepsilon > 0$,

$$C^{\frac{1}{2}-H+\varepsilon}(T) \subset \mathcal{H} \subset L^2(T) \subset L^{\frac{1}{H}}(T). \tag{6}$$

Consider $\mathcal{E}_{\mathcal{H}}$ the class of step functions of the form

$$\varphi(\cdot) = \sum_{i=1}^{n} a_i 1_{(t_i,t_{i+1}]}(\cdot) \qquad n \geq 1, t_i \in T, a_i \in \mathbb{R}. \tag{7}$$

It has been proved in [14] that $\mathcal{E}_{\mathcal{H}}$ is dense in $\mathcal{H}$. For $\varphi \in \mathcal{E}_{\mathcal{H}}$ of the form (7) we define its Wiener integral with respect to the fBm B^H by

$$\int_0^1 \varphi(s) dB_s^H := \sum_{i=1}^{n} a_i \left(B_{t_{i+1}}^H - B_{t_i}^H\right) \tag{8}$$

The mapping $\varphi \to \int_0^1 \varphi(s) dB_s^H$ provides an isometry between $\mathcal{E}_{\mathcal{H}}$ and the first chaos of the fBm B^H and it can be extended as follows:

- If $H > \frac{1}{2}$, it has been proved in [13] that $\mathcal{E}_{\mathcal{H}}$ is dense in $L_H^2(T)$ with respect to the norm $\|\cdot\|_H$. As a consequence, the Wiener integral $\int_0^1 \varphi(s) dB_s^H$ can be defined in a consistent way as limit in $L^2(\Omega)$ of integrals of elementary functions for any $\varphi \in L_H^2(T)$.

- If $H < \frac{1}{2}$, then $\mathcal{E}_{\mathcal{H}}$ is dense in $\mathcal{H}$ (see [8], [13]) and the integral $\int_0^1 \varphi(s) dB_s^H$ can be defined by isometry for any function $\varphi \in \mathcal{H}$.

We will need in this paper stochastic integrals of the form $\int_T u_s dB_s^H$ where u is a stochastic process independent by B^H. Using the above facts, it follows that this integral can be defined by isometry for any $u \in L^2(\Omega) \times L_H^2(T)$ if $H > \frac{1}{2}$ and for any $u \in L^2(\Omega;\mathcal{H})$ if $H < \frac{1}{2}$.

Remark 1 *The integral $\int_T u_s dB_s^H$ coincides also with the Skorohod integral introduced in [2], [1] since, by independence, the Malliavin derivative of u with respect to B^H is zero.*

More generally, for $H > \frac{1}{2}$, let $L_H^2(T^n)$ be the set of functions $f : T^n \to \mathbb{R}$ such that

$$\int_{T^n} |f(u_1,\ldots,u_n)||f(v_1,\ldots,v_n)| \left(\prod_{i=1}^{n} \theta^H(u_i,v_i)\right) du_1 \ldots du_n dv_1 \ldots dv_n < \infty,$$

endowed with the scalar product

$$\langle f, g\rangle_{H^n} = \int_{T^n} f(u_1, \ldots, u_n)\, g(v_1, \ldots, v_n) \left(\prod_{i=1}^{n} \theta^H(u_i, v_i)\right) du_1 \ldots du_n dv_1 \ldots dv_n. \tag{9}$$

Obviously, $L^2_H(T^n)$ is a subset of $\mathcal{H}^{\otimes n}$ and if $f, g \in L^2_H(T^n)$ then we have

$$\langle f, g\rangle_{H^n} = \langle f, g\rangle_{\mathcal{H}^{\otimes n}}.$$

We will denote by $L^2_{s,H}(T^n)$ the set of symmetric functions $f \in L^2_H(T^n)$ and if $f \in L^2_{s,H}(T^2)$ let us introduce the (Hilbert-Schmidt) operator (see [7]) $K^H_f : L^2_H(T) \to L^2_H(T)$ given by

$$\left(K^H_f \varphi\right)(y) = \int_T \int_T f(x, y)\varphi(x')\theta^H(x, x')dxdx'. \tag{10}$$

Remark 2 *Note that if f is positive and $H > \frac{1}{2}$, then the operator K^H_f is a positive operator. Indeed, we can write*

$$\left(K^H_f \varphi\right)(y) = \int_T A(x', y)\varphi(x')dx'$$

where $A(x', y) = \int_T f(x, y)\theta^H(x, x')dx$ is positive. Thus the eigenvalues of K^H_f are positive.

3 The characteristic function of the double integral

Throughout this section B^H and B^α will denote two independent fractional Brownian motion with parameter H and α respectively. We compute the characteristic function of the random variable

$$S := \int_T B^\alpha_s dB^H_s. \tag{11}$$

Note that, when $H > \frac{1}{2}$, the random variables S (11) is well-defined since obviously B^α belongs to $L^2(\Omega) \times L^2_H(T)$ for any α. When $H < \frac{1}{2}$, if we assume that $\alpha + H > \frac{1}{2}$, then we have $B^\alpha \in C^{\frac{1}{2}-H+\varepsilon}(T)$. But, in the following we will need to restrict ourselves to the situation $H > \frac{1}{2}$.

We start with the following lemma which gives an approximation of the random variable S given by (11) when the Hurst parameter of the integrator fbm B^H is bigger than one half.

Lemma 1 *Assume that $H > \frac{1}{2}$ and $\alpha \in (0, 1)$. Denote by*

$$T_n = \sum_{i=0}^{n-1} B^\alpha_{t_i}\left(B^H_{t_{i+1}} - B^H_{t_i}\right) \tag{12}$$

where $\pi : 0 = t_0 < t_1 < \ldots < t_n = 1$ denotes a partition of $[0, 1]$. Then it holds that

$$T_n \to S \text{ in } L^2(\Omega) \text{ as } |\pi| \to 0.$$

Proof: Using the independence of B^α and B^H we can write

$$B^\alpha_{t_i}\left(B^H_{t_{i+1}} - B^H_{t_i}\right) = \int_{t_i}^{t_{i+1}} B^\alpha_{t_i} dB^H_s.$$

To prove the lemma it is enough to prove that

$$\sum_{i=0}^{n-1} B^\alpha_{t_i} 1_{[t_i,t_{i+1}]}(\cdot) \to B^\alpha_\cdot = \sum_{i=0}^{n-1} B^\alpha_\cdot 1_{[t_i,t_{i+1}]}(\cdot) \text{ in } L^2(\Omega) \times L^2_H(T) \text{ as } |\pi| \to 0.$$

Actually in general, to prove the convergence of a sequence of stochastic integrals of divergence type one needs also the convergence of the Malliavin derivatives, but in our case it is unnecessary due to the independence of the two fBms. We have, using formula (4),

$$\begin{aligned}
& E\left\|\sum_{i=0}^{n-1}\left(B^\alpha_{t_i} - B^\alpha_\cdot\right) 1_{[t_i,t_{i+1}]}\right\|^2_H \\
&= \sum_{i,j=0}^{n-1} H(2H-1)\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} E\left(B^\alpha_{t_i} - B^\alpha_s\right)\left(B^\alpha_{t_j} - B^\alpha_r\right)|r-s|^{2H-2} dr ds \\
&\le \sum_{i,j=0}^{n-1} H(2H-1)\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} |t_i - s|^\alpha |t_j - r|^\alpha |r-s|^{2H-2} dr ds \\
&\le H(2H-1)|\pi|^{2\alpha}\sum_{i,j=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} |r-s|^{2H-2} dr ds \\
&= |\pi|^{2\alpha}\sum_{i,j=0}^{n-1}\langle 1_{[t_i,t_{i+1}]}, 1_{[t_j,t_{j+1}]}\rangle_H = |\pi|^{2\alpha}
\end{aligned}$$

and this goes to 0 for every $\alpha \in (0,1)$. ■

We will also need to prove the following technical lemma:

Lemma 2 *a) Assume that $\alpha > \frac{1}{2}$ and consider the function*

$$f^H(x,y) = \frac{1}{2}\left((1-x)^{2H} + (1-y)^{2H} - |x-y|^{2H}\right), \qquad x,y \in [0,1]. \tag{13}$$

Then $f^H \in L^2_{s,\alpha}(T^2)$.

b) Assume that $H > \frac{1}{2}$ and consider the function

$$f^\alpha(x,y) = \frac{1}{2}\left(x^{2\alpha} + y^{2\alpha} - |x-y|^{2\alpha}\right). \tag{14}$$

Then $f^\alpha \in L^2_{s,H}(T^2)$.

Proof: Let us prove first the point 2a); the point 2b) is similar. We have to show that

$$I := \int_0^1 \int_0^1 \int_0^1 \int_0^1 f^H(x_1, y_1) f^H(x_2, y_2) \theta^\alpha(x_1, x_2) \theta^\alpha(y_1, y_2) dx_1 dx_2 dy_1 dy_2 < \infty.$$

Note that

$$\begin{aligned} \left| f^H(x_i, y_i) \right| &= E\left(B_1^H - B_{x_i}^H\right)\left(B_1^H - B_{y_i}^H\right) \\ &\leq \left(E\left(B_1^H - B_{x_i}^H\right)^2\right)^{1/2} \left(E\left(B_1^H - B_{y_i}^H\right)^2\right)^{1/2} \\ &= (1 - x_i)^H (1 - y_i)^H \end{aligned}$$

The integral I is therefore bounded by

$$\begin{aligned} I &\leq (c(\alpha))^2 \int_{[0,1]^4} (1 - x_1)^H (1 - y_1)^H (1 - x_2)^H (1 - y_2)^H |x_1 - x_2|^{2\alpha-2} |y_1 - y_2|^{2\alpha-2} dx_1 dx_2 dy_1 dy_2 \\ &= \left(c(\alpha) \int_0^1 \int_0^1 (1 - x_1)^H (1 - x_2)^H |x_1 - x_2|^{2\alpha-2} dx_1 dx_2 \right)^2 \end{aligned}$$

with $c(\alpha) = \alpha(2\alpha - 1)$. Now, using the change of variables $z = \frac{x-y}{1-y}$, we get

$$\begin{aligned} I' &:= \int_0^1 \int_0^1 (1 - x)^H (1 - y)^H |x - y|^{2\alpha-2} dy dx \\ &= 2 \int_0^1 \int_0^x (1 - x)^H (1 - y)^H (x - y)^{2\alpha-2} dy dx \\ &= 2 \int_0^1 (1 - x)^{2H+2\alpha-1} \left(\int_0^x (1 - z)^{-H-2\alpha} z^{2\alpha-2} dz \right) dx \\ &= \frac{1}{H + \alpha} \int_0^1 (1 - z)^H z^{2\alpha-2} dz < \infty, \end{aligned}$$

using that $\alpha > \frac{1}{2}$. ■

We state now our main result.

Theorem 1 *Let $\alpha > \frac{1}{2}$ and $H > \frac{1}{2}$. Then the characteristic function of the random variable S given by (11) is*

$$E\left(e^{itS}\right) = \prod_{i \geq 1} \left(\frac{1}{1 + t^2 \mu_i} \right)^{\frac{1}{2}}$$

where $(\mu_i)_{i \geq 1}$ are the eigenvalues of the operator $K^\alpha_{f^H}$ given by (10) where f^H is defined by (13).

Remark 3 *If $\alpha = \frac{1}{2}$, then the operator $K^{\alpha}_{f^H}$ must be replaced by*

$$\left(K^{\frac{1}{2}}_{f^H}\varphi\right)(y) = \int_0^1 f^H(x,y)\varphi(x)dx. \tag{15}$$

Proof of Theorem 1: By Lemma 1 we have

$$E\left(e^{itT}\right) = \lim_{n\to\infty} E\left(e^{itT_n}\right)$$

where T_n is given by (12) with $t_i = \frac{i}{n}$, for every $i = 0,\ldots,n-1$. Let us compute the characteristic function of the random variable T_n.

We will use the following fact: If X, Y are two independent random variables, then

$$E\left(\Phi(X,Y)/X\right) = \varphi(X)$$

where $\varphi(x) = E(\Phi(x,Y))$. Let us put

$$X = \left(B^{\alpha}_0, B^{\alpha}_1, \ldots, B^{\alpha}_{\frac{n-1}{n}}\right) \text{ and } Y = \left(B^H_{\frac{1}{n}} - B^H_0, \ldots, B^H_{\frac{n}{n}} - B^H_{\frac{n-1}{n}}\right). \tag{16}$$

Therefore, we obtain

$$\varphi(x) = E\left(e^{it\sum_{k=0}^{n-1} x_k Y_k}\right) = e^{-\frac{t^2}{2}x^T A^H x}$$

where the matrix $A^H = \left(A^H_{k,l}\right)_{k,l=0,\ldots,n-1}$ is given by

$$\begin{aligned} A^H_{k,l} &= E\left(B^H_{\frac{k+1}{n}} - B^H_{\frac{k}{n}}\right)\left(B^H_{\frac{l+1}{n}} - B^H_{\frac{l}{n}}\right) \\ &= \frac{1}{2n^{2H}}\left(|k-l+1|^{2H} + |k-l-1|^{2H} - 2|k-l|^{2H}\right). \end{aligned}$$

We will obtain

$$E\left(e^{itT_n}\right) = E(e^{-\frac{t^2}{2}S_n})$$

where

$$\begin{aligned} S_n &:= \sum_{k,l=0}^{n-1} A^H_{k,l} B^{\alpha}_{\frac{k}{n}} B^{\alpha}_{\frac{l}{n}} \\ &= \sum_{k,l=1}^{n-1} A^H_{k,l} B^{\alpha}_{\frac{k}{n}} B^{\alpha}_{\frac{l}{n}} \\ &= \sum_{k,l=1}^{n-1} A^H_{k,l}\left(\sum_{k'=0}^{k-1}\left(B^{\alpha}_{\frac{k'+1}{n}} - B^{\alpha}_{\frac{k'}{n}}\right)\right)\left(\sum_{l'=0}^{l-1}\left(B^{\alpha}_{\frac{l'+1}{n}} - B^{\alpha}_{\frac{l'}{n}}\right)\right) \\ &= \sum_{k',l'=0}^{n-2}\left(B^{\alpha}_{\frac{k'+1}{n}} - B^{\alpha}_{\frac{k'}{n}}\right)\left(B^{\alpha}_{\frac{l'+1}{n}} - B^{\alpha}_{\frac{l'}{n}}\right)\sum_{l=l'+1}^{n-1}\sum_{k=k'+1}^{n-1} A^H_{k,l}. \end{aligned} \tag{17}$$

We calculate first

$$
\begin{aligned}
&\sum_{l=l'+1}^{n-1}\sum_{k=k'+1}^{n-1} A^H_{k,l} \\
= \;& \frac{1}{2n^{2H}}\sum_{l=l'+1}^{n-1}\left[\sum_{k=k'+1}^{n-1}\left(|k-l+1|^{2H}+|k-l-1|^{2H}-2|k-l|^{2H}\right)\right] \\
= \;& \frac{1}{2n^{2H}}\sum_{l=l'+1}^{n-1}\left[\sum_{k=k'+1}^{n-1}\left(|k-l+1|^{2H}-|k-l|^{2H}\right)-\sum_{k=k'+1}^{n-1}\left(|k-l|^{2H}-|k-l-1|^{2H}\right)\right] \\
= \;& \frac{1}{2n^{2H}}\sum_{l=l'+1}^{n-1}\left[|n-l|^{2H}-|k'+1-l|^{2H}-|n-1-l|^{2H}+|k'-l|^{2H}\right] \\
= \;& \frac{1}{2n^{2H}}\left[\sum_{l=l'+1}^{n-1}\left(|l-k'|^{2H}-|l-1-k'|^{2H}\right)-\sum_{l=l'+1}^{n-1}\left(|l+1-n|^{2H}-|l-n|^{2H}\right)\right] \\
= \;& \frac{1}{2n^{2H}}\left[(n-k'-1)^{2H}+(n-l'-1)^{2H}-|l'-k'|^{2H}\right] \\
= \;& f^H\left(\frac{k'+1}{n},\frac{l'+1}{n}\right) \qquad (18)
\end{aligned}
$$

where the function f^H is given by (13). By (17) and (18) we get

$$
S_n=\sum_{k,l=0}^{n-1} f^H\left(\frac{k+1}{n},\frac{l+1}{n}\right)\left(B^\alpha_{\frac{k+1}{n}}-B^\alpha_{\frac{k}{n}}\right)\left(B^\alpha_{\frac{l+1}{n}}-B^\alpha_{\frac{l}{n}}\right).
$$

Let us denote by $(\mu_i)_{i\geq 1}$ the eigenvalues of the operator $K^\alpha_{f^H}$ and by $(g_i)_{i\geq 1}$ the corresponding eigenfunctions. Then, using Lemma 2, we can write

$$
f^H(x,y)=\sum_{i\geq 1}\mu_i g_i(x)g_i(y)
$$

with the vectors $(g_i)_{i\geq 1}$ orthogonal in $L^2_{s,\alpha}(T)$ and the μ_i are square-summable.

The sum S_n becomes

$$
\begin{aligned}
S_n \;&=\; \sum_{k,l=0}^{n-1}\left(\sum_{i\geq 1}\mu_i g_i(\frac{k+1}{n})g_i(\frac{l+1}{n})\right)\left(B^\alpha_{\frac{k+1}{n}}-B^\alpha_{\frac{k}{n}}\right)\left(B^\alpha_{\frac{l+1}{n}}-B^\alpha_{\frac{l}{n}}\right) \\
&=\; \sum_{i\geq 1}\mu_i\left(\sum_{k=0}^{n-1} g_i(\frac{k+1}{n})\left(B^\alpha_{\frac{k+1}{n}}-B^\alpha_{\frac{k}{n}}\right)\right)^2.
\end{aligned}
$$

Since $\alpha>\frac{1}{2}$ and $g_i\in L^2_{s,\alpha}(T)$ it follows from [13] that

$$
\sum_{k=0}^{n-1} g_i(\frac{k+1}{n})\left(B^\alpha_{\frac{k+1}{n}}-B^\alpha_{\frac{k}{n}}\right)\overset{|\pi|\to 0}{\longrightarrow}\int_0^1 g_i(x)dB^\alpha(x) \text{ in } L^2(\Omega)
$$

and therefore we have that

$$S_n \overset{n\to\infty}{\longrightarrow} \sum_{i\geq 1} \mu_i H_i^2 \qquad \text{in } L^1(\Omega)$$

where $\left(H_i = \int_0^1 g_i(x) dB^\alpha(x), i \geq 1\right)$ are independent, standard normal random variables. As a consequence, since the eigenvalues are positive (see Remark 1)

$$\begin{aligned} E(e^{itT}) &= E\left(\exp\left(-\frac{t^2}{2}\sum_{i\geq 1}\mu_i H_i^2\right)\right) \\ &= \prod_{i\geq 1} E\left(\exp\left(-\frac{t^2}{2}\mu_i H_i^2\right)\right) \\ &= \prod_{i\geq 1}\left(\frac{1}{1+t^2\mu_i}\right)^{\frac{1}{2}}. \end{aligned}$$

■

We can state an alternative result that allows to consider the situation when the Hurst parameter of the integrand α is less than $\frac{1}{2}$.

Theorem 2 *Assume that $H > \frac{1}{2}$ and $\alpha \in (0,1)$. Then the characteristic function of S (11) is*

$$E\left(e^{itS}\right) = \prod_{i\geq 1}\left(\frac{1}{1+t^2\mu_i}\right)^{\frac{1}{2}}$$

where $(\mu_i)_{i\geq 1}$ are the eigenvalues of the operator $K^H_{f^\alpha}$ given by (10) and f^α is defined by (14).

Proof: We follow the lines of Theorem 1 by interchanging the roles of X and Y in (16). We obtain that $E(e^{itS}) = \lim_{n\to\infty} E\left(e^{-\frac{t^2}{2}S_n}\right)$ where

$$\begin{aligned} S_n &= \sum_{k,l=0}^{n-1} E\left(B^\alpha_{\frac{k}{n}} B^\alpha_{\frac{l}{n}}\right)\left(B^H_{\frac{k+1}{n}} - B^H_{\frac{k}{n}}\right)\left(B^H_{\frac{l+1}{n}} - B^H_{\frac{l}{n}}\right) \\ &= \sum_{k,l=0}^{n-1} f^\alpha\left(\frac{k}{n},\frac{l}{n}\right)\left(B^H_{\frac{k+1}{n}} - B^H_{\frac{k}{n}}\right)\left(B^H_{\frac{l+1}{n}} - B^H_{\frac{l}{n}}\right) \end{aligned}$$

where f^α is given by (14). Now we use Lemma 2b) and we proceed as in the proof of Theorem 1. ■

4 The two-parameter case

In this section, we will briefly discuss the case of the fractional Brownian sheet. Let us denote by $\left(B^{\alpha_1,\alpha_2}_{s,t}\right)_{s,t\in T}$ and $\left(B^{H_1,H_2}_{s,t}\right)_{s,t\in T}$ two independent fractional Brownian sheets. We recall that a fractional Brownian sheet $\left(B^{H_1,H_2}_{s,t}\right)_{s,t\in T}$ with Hurst parameters $H_1, H_2 \in (0,1)$ is a centered Gaussian process starting from 0 with covariance given by

$$E\left(B^{H_1,H_2}_{s,t} B^{H_1,H_2}_{u,v}\right) = R^{H_1}(s,u)R^{H_2}(t,v), \qquad s,t,u,v \in T,$$

where R^{H_i} is the covariance of the one-parameter fBm with Hurst index H_i $(i = 1,2)$. We refer to [4] or [3] for the basic properties and [15], [16] or [10] for elements of the stochastic calculus with respect to this process. We only point here the following facts:

- the canonical Hilbert space $\mathcal{H}(H_1,H_2)$ of the Gaussian process B^{H_1,H_2} is defined as the closure of the linear vector space generated by the indicator functions $\{1_{[0,s]\times[0,t]}, s,t \in T\}$ with respect to the scalar product

$$\langle 1_{[0,s]\times[0,t]}, 1_{[0,u]\times[0,v]}\rangle_{\mathcal{H}(H_1,H_2)} = R^{H_1}(s,u)R^{H_2}(t,v).$$

- if H_1 or H_2 is bigger than $\frac{1}{2}$, then the elements of $\mathcal{H}(H_1,H_2)$ maybe not functions but distributions. In this case it is convenient to work with the following subspace of $\mathcal{H}(H_1,H_2)$

$$L^2_{H_1,H_2}\left(T^2\right) := L^2_{H_1}(T) \otimes L^2_{H_2}(T)$$

  which is a space of functions (and which plays the role played by $L^2_H(T)$ in the one-parameter case). Therefore Wiener integrals with respect to B^{H_1,H_2} can be naturally defined for integrands in $L^2_{H_1,H_2}\left(T^2\right)$.

We prove here the following result.

Theorem 3 *Assume that $H_i > \frac{1}{2}$ and $\alpha_i > \frac{1}{2}$, $i = 1,2$. Then the characteristic function of the random variable*

$$A := \int_T \int_T B^{\alpha_1,\alpha_2}_{u,v} dB^{H_1,H_2}_{u,v} \tag{19}$$

is given by

$$E\left(e^{itA}\right) = \prod_{i,j\geq 1} \left(\frac{1}{1+t^2\mu_{i,1}\mu_{j,2}}\right)^{\frac{1}{2}} \tag{20}$$

where $(\mu_{i,1})_i$ are the eigenvalues of the operator $K^{\alpha_1}_{f^{H_1}}$ given by (10), $(\mu_{j,2})_j$ are the eigenvalues of $K^{\alpha_2}_{f^{H_2}}$ and f^{H_1}, f^{H_2} are defined by (13).

Proof: Denote by

$$A_n := \sum_{k,l=0}^{n-1} B^{\alpha_1,\alpha_2}_{\frac{k}{n},\frac{l}{n}} B^{H_1,H_2}(\Delta_{k,l})$$

where

$$B^{H_1,H_2}(\Delta_{k,l}) = B^{H_1,H_2}_{\frac{k+1}{n},\frac{l+1}{n}} - B^{H_1,H_2}_{\frac{k}{n},\frac{l+1}{n}} - B^{H_1,H_2}_{\frac{k+1}{n},\frac{l}{n}} + B^{H_1,H_2}_{\frac{k}{n},\frac{l}{n}}$$

As in Lemma 1, we can prove that $A_n \to A$ when $n \to \infty$ in $L^2(\Omega)$ for $\alpha_i > \frac{1}{2}$, $H_i > \frac{1}{2}$, $i = 1, 2$. We obtain, using the methods used in the proof of Lemma 1 (see also [9]) that

$$E(e^{itA}) = \lim_{n\to\infty} E\left(e^{itS_n}\right)$$

with

$$S_n = \sum_{k,l=0}^{n-1} \sum_{k',l'=0}^{n-1} f^{H_1}\left(\frac{k+1}{n}, \frac{k'+1}{n}\right) f^{H_2}\left(\frac{l+1}{n}, \frac{l'+1}{n}\right) B^{\alpha_1,\alpha_2}(\Delta_{k,l}) B^{\alpha_1,\alpha_2}(\Delta_{k',l'}).$$

By Lemma 2a) we get that $f^{H_i} \in L^2_{s,\alpha_i}(T)$ $(i = 1, 2)$ and thus $f_{H_i} = \sum_k \mu_{k,i} g_{k,i}$ where $(g_{k,i})_{k\geq 1}$ are the eigenvectors of $K^{\alpha_i}_{f^{H_i}}$ $(i = 1, 2)$.

$$S_n = \prod_{i,j\geq 1} \mu_{i,1}\mu_{j,2} \left(\sum_{k,l=0}^{n-1} g_{i,1}(\frac{k+1}{n}) g_{j,2}(\frac{l+1}{n}) B^{\alpha_1,\alpha_2}(\Delta_{k,l}) \right)^2.$$

Since $g_{i,1} \in L^2_{\alpha_1}(T)$ for every $i \geq 1$ and $g_{j,2} \in L^2_{\alpha_2}(T)$ for every $j \geq 1$, we have that $g_{i,1} \otimes g_{j,2} \in L^2_{\alpha_1,\alpha_2} 9T^2$ and it is not difficult to see that

$$\sum_{k,l=0}^{n-1} g_i(\frac{k+1}{n}) g_j(\frac{l+1}{n}) B^{\alpha_1,\alpha_2}(\Delta_{k,l}) \to_{n\to\infty} \int_T \int_T g_i(x) g_j(y) dB^{\alpha_1,\alpha_2}_{x,y} := H_{i,j}$$

and the random variables $H_{i,j}$ are mutually independent and $N(0, 1)$ distributed. The conclusion follows easily. ■

Remark 4 *If $H_i > \frac{1}{2}$ and $\alpha_i \in (0, 1)$, we obtain*

$$E\left(e^{itA}\right) = \prod_{i,j\geq 1} \left(\frac{1}{1 + t^2 \mu_{i,1}\mu_{j,2}} \right)^{\frac{1}{2}}$$

where for $j = 1, 2$, $(\mu_{i,j})_i$ are the eigenvalues of the operator $K^{H_j}_{f^{\alpha_j}}$, where f^{α_j} is defined by (14).

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