On the second moment of the number of crossings by a stationary Gaussian process

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Abstract

Cramér and Leadbetter introduced in 1967 the sufficient condition

$$\int_0^\delta \frac{r''(s) - r''(0)}{s} ds < \infty, \qquad \delta > 0,$$

to have a finite variance of the number of zeros of a centered stationary Gaussian process with twice differentiable covariance function r. This condition is known as the Geman condition, since Geman proved in 1972 that it was also a necessary condition. Up to now no such criterion was known for counts of crossings of a level other than the mean. This paper shows that the Geman condition is still sufficient and necessary to have a finite variance of the number of any fixed level crossings. For the generalization to the number of a curve crossings, a condition on the curve has to be added to the Geman condition.

1 Introduction and main result

Let $X = \{X_t,\ t \in R\}$ be a centered stationary Gaussian process with covariance r and spectral measure μ. The function r can be expressed as $r(t) = \int_{-\infty}^{\infty} e^{it\lambda}\mu(d\lambda)$, and is supposed to be twice differentiable.
Let consider a continuous differentiable real function ψ and let define, as in Cramér & Leadbetter ([3]), the number of crossings of the function ψ by the process X on an interval $[0, t]$ ($t \in \mathbb{R}$), as the random variable

$$N_t^\psi = N_t(\psi) = \#\{s \le t : X_s = \psi_s\}.$$

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N_t^ψ can also be seen as the number of zero crossings $N_t^Y(0)$ by the non-stationary Gaussian process $Y = \{Y_s,\ s \in R\}$, with $Y_s := X_s - \psi_s$, i.e.

$$N_t^\psi = N_t^Y(0).$$

Note that Y is non-stationary, but stationary in the sense of the covariance, since it has the same covariance function as X.

On what concerns the moments of the number of crossings by X, we can recall one of the most well-known first results obtained by Rice in 1945 (cf. [10]) for a given level x, namely

$$\mathrm{E}[N_t(x)] = \ t e^{-x^2/2} \sqrt{-r''(0)}\ /\pi.$$

Two decades later, Itô ([8], 1964) and Ylvisaker ([12], 1965) provided a necessary and sufficient condition to have a finite mean number of crossings:

$$\mathrm{E}[N_t(x)] < \infty \quad \Leftrightarrow \quad \lambda_2 < \infty \quad \Leftrightarrow \quad -r''(0) < \infty.$$

Also in the 60's, following on the work of Cramér, generalization to curve crossings and higher order moments for $N_t(.)$ were considered in a series of papers by Cramér and Leadbetter and Ylvisaker.
A generalized Rice formula was proposed by Ylvisaker (1966, [13]) and Cramér & Leadbetter (1967, [3]), when considering the number of crossings of ψ:

$$\mathrm{E}[N_t(\psi)] = \sqrt{-r''(0)} \int_o^t \varphi(\psi(y)) \left[2\varphi\left(\frac{\psi'(y)}{\sqrt{-r''(0)}}\right) + \frac{\psi'(y)}{\sqrt{-r''(0)}} \left(2\Phi\left(\frac{\psi'(y)}{\sqrt{-r''(0)}}\right) - 1 \right) \right] dy,$$

where φ and Φ are respectively the standard normal density and distribution function.
Concerning the second factorial moment, an explicit formula for the number of zeros of the process X was given in Cramér &Leadbetter (see [3], pp. 209), from which can be deduced the following formula for the second factorial moment of the number of crossings of the function ψ by X:

$$M_2^\psi = \int_0^t \int_0^t \int_{R^2} |\dot{x}_1 - \dot{\psi}_{t_1}||\dot{x}_2 - \dot{\psi}_{t_2}| p_{t_1,t_2}(\psi_{t_1}, \dot{x}_1, \psi_{t_2}, \dot{x}_2) d\dot{x}_1 d\dot{x}_2 \ dt_1 dt_2 \ , \tag{1}$$

where $p_{t_1,t_2}(x_1, \dot{x}_1, x_2, \dot{x}_2)$ is the density of the vector $(X_{t_1}, \dot{X}_{t_1}, X_{t_2}, \dot{X}_{t_2})$ that is supposed non-singular for all $t_1 \neq t_2$. The formula holds whether M_2^ψ is finite or not.
Cramér and Leadbetter proposed in 1967 a sufficient condition on the correlation function of X in order to have the random variable $N_t(0)$ belonging to $L^2(\Omega)$, namely

$$\text{If} \quad L(t) := \frac{r''(t) - r''(0)}{t} \ \in L^1([0,\delta], dx)$$
$$\text{then} \qquad \mathrm{E}[N_t^2(0)] < \infty.$$

This last condition is known as **Geman condition**, since Geman proved in 1972 that it was not only sufficient but also necessary:

$$L(t) = \frac{r''(t) - r''(0)}{t} \in L^1([0,\delta], dx) \quad \Leftrightarrow \quad \mathrm{E}[N_t^2(0)] < \infty. \tag{2}$$

Note that this condition held only when choosing the level as the mean of the process. Generalizing this result to any given level x and to some differentiable curve ψ has been the object of some investigation and we could mention some nice papers, as for instance the ones of Cuzick ([5], [6]), proposing sufficient conditions. But to get necessary conditions remained an open problem for many years.
The purpose of this paper is to solve this problem; the solution is enunciated in the following theorem.

Theorem

1) *For any given level x, we have*

$$\mathrm{E}[N_t^2(x)] < \infty \quad \Leftrightarrow \quad L(t) = \frac{r''(t) - r''(0)}{t} \in L^1([0,\delta], dx) \quad (\textit{Geman condition}).$$

2) *Suppose that the continuous differentiable real function ψ satisfies*

$$\dot\psi(t+\tau) = \dot\psi(t) + \gamma(t,\tau), \quad \textit{for} \ \ \tau \in [0,\delta], \ 0 \le \delta < 1,$$

and

$$\int_0^\delta \frac{\tilde\gamma(s)}{s} ds < \infty, \quad \textit{where } \tilde\gamma(\tau) \textit{ is the modulus of continuity of } \dot\psi.$$

Then

$$\mathrm{E}[N_t^2(\psi)] < \infty \quad \Leftrightarrow \quad L(t) \ \in L^1([0,\delta], dx).$$

The method used to prove that the Geman condition keeps being the sufficient and necessary condition to have a second moment finite in these different cases, is quite simple. It relies on the study of some functions of r and its derivatives at the neighborhood of 0, and the chaos expansion of the second moment.
Finally let us mention the work of Belyaev ([1]), and Cuzick ([4], [5] and [6]) who proposed some sufficient conditions to have the finiteness of the kth (factorial) moments for the number of crossings for $k \ge 2$. When $k \ge 3$, the difficult problem of finding necessary conditions when considering levels other than the mean is still open.

2 Study of the second moment

Let us give another formula for the second factorial moment M_2^ψ given in (1).
First we compute

$$\begin{aligned} I(t_1,t_2) &:= \int_{R^2} |\dot x_1 - \dot\psi_{t_1}||\dot x_2 - \dot\psi_{t_2}| p_{t_1,t_2}(\psi_{t_1}, \dot x_1, \psi_{t_2}, \dot x_2) d\dot x_1 d\dot x_2 \\ &= p_{t_1,t_2}(\psi_{t_1}, \psi_{t_2}) \,\mathrm{E}\left[|\dot X_{t_1} - \dot\psi_{t_1}||\dot X_{t_2} - \dot\psi_{t_2}| \mid X_{t_1} = \psi_{t_1}, X_{t_2} = \psi_{t_2}\right], \end{aligned} \tag{3}$$

where $p_{t_1,t_2}(x_1,x_2)$ is the density of vector (X_{t_1},X_{t_2}).
Notice that $I(t_1,t_2)=I(t_2,t_1)$, so that we can write

$$M_2^\psi=\int_0^t\int_{t_1}^t I(t_1,t_2)dt_2dt_1+\int_0^t\int_{t_2}^t I(t_1,t_2)dt_1dt_2=2\int_0^t\int_{t_1}^t I(t_1,t_2)dt_2dt_1. \qquad (4)$$

Hence from now on, we put $t_2=t_1+\tau$, $\tau>0$.
We will be using the following regression model:

$$(R)\qquad \begin{cases} \dot X_{t_1}=\zeta+\alpha_1(\tau)X_{t_1}+\alpha_2(\tau)X_{t_1+\tau}\\ \dot X_{t_1+\tau}=\zeta^*-\beta_1(\tau)X_{t_1}-\beta_2(\tau)X_{t_1+\tau}\end{cases}$$

where (ζ,ζ^*) is jointly Gaussian such that

$$Var(\zeta)=Var(\zeta^*):=\sigma^2(\tau)=-r''(0)-\frac{r'^{\,2}(\tau)}{1-r^2(\tau)}, \qquad (5)$$

$$Cov(\zeta,\zeta^*)=-r''(\tau)-\frac{r'^{\,2}(\tau)r(\tau)}{1-r^2(\tau)},$$

$$\rho(\tau):=\frac{Cov(\zeta,\zeta^*)}{\sigma^2(\tau)}=\frac{-r''(\tau)\left(1-r^2(\tau)\right)-r'^2(\tau)r(\tau)}{-r''(0)\left(1-r^2(\tau)\right)-r'^2(\tau)}, \qquad (6)$$

and where $\begin{cases}\alpha_1=\alpha_1(\tau)=\frac{r'(\tau)r(\tau)}{1-r^2(\tau)}\\ \alpha_2=\alpha_2(\tau)=-\frac{r'(\tau)}{1-r^2(\tau)}\\ \beta_1=\beta_1(\tau)=\alpha_2(\tau)\\ \beta_2=\beta_2(\tau)=\alpha_1(\tau).\end{cases}$

Note that $\alpha_1+\alpha_2=\beta_1+\beta_2$.

We will mainly work in the neighborhood of 0, that's why we will study the behavior of some functions on this neighborhood.
Suppose that the correlation function r satisfies on $[0,\delta]$, $\delta>0$,

$$\begin{cases} r(\tau) &= 1-\frac{-r''(0)}{2}\tau^2+\theta(\tau), \quad \theta(\tau)>0,\\ r'(\tau) &= -(-r''(0))\tau+\theta'(\tau),\\ r''(\tau) &= -(-r''(0))+\theta''(\tau),\end{cases} \qquad (7)$$

with $\dfrac{\theta(\tau)}{\tau^2}\to 0$, $\dfrac{\theta'(\tau)}{\tau}\to 0$ and $\theta''\to 0$ as $\tau\to 0$.
Let us introduce the nonnegative function L such that

$$\theta''(\tau):=\tau L(\tau),$$

then $\theta'(\tau) = \int_0^\tau uL(u)du$ and $\theta(\tau) = \int_0^\tau \int_0^v uL(u)dudv$.

In all what follows, the notation $f(\tau) \sim g(\tau)$ means $\frac{f(\tau)}{g(\tau)} \to C > 0$ as $\tau \to 0$.

On a neighborhood of 0, we have

$$\alpha_2(\tau) \sim \frac{1}{\tau}, \qquad \alpha_1(\tau) \sim -\alpha_2(\tau), \tag{8}$$

$$\sigma^2(\tau) \sim 2\left(\frac{\theta'(\tau)}{\tau} - \frac{\theta(\tau)}{\tau^2}\right) \tag{9}$$

and

$$\rho(\tau) \sim 1 - \frac{\theta''(\tau)}{2\left(\frac{\theta'(\tau)}{\tau} - \frac{\theta(\tau)}{\tau^2}\right)}. \tag{10}$$

Let μ_4 denote the fourth spectral moment of μ, i.e. $\mu_4 := \int_{-\infty}^{\infty} \lambda^4 d\mu(\lambda)$.

We introduce now three lemmas useful to prove the Theorem, but which have some interests on their own. Indeed, Lemmas 1 and 3 show that the behavior of the Geman function L is closely related to the existence of μ_4 or to the behavior of the variance of the r.v. ζ (introduced in the regression model (R)), respectively, whereas Lemma 2 provides some study on the correlation function ρ of the r.v. ζ and on the function r' in the neighborhood of 0.

Lemma 1

(i) If $\mu_4 = +\infty$*, then* $\lim_{\tau\to 0} \frac{L(\tau)}{\tau} = +\infty$*, or equivalently* $\lim_{\tau\to 0} \frac{\tau}{L(\tau)} = 0$.

(ii) If $\mu_4 < +\infty$*, then* $\lim_{\tau\to 0} \frac{L(\tau)}{\tau} = \int_0^\infty \lambda^4 d\mu(\lambda) = \frac{r^{iv}(0)}{2}$*, or* $\lim_{\tau\to 0} \frac{\tau}{L(\tau)} = \frac{2}{r^{iv}(0)}$.

Remark: This lemma could also be formulated as

$$\lim_{\tau\to 0} \frac{\tau}{L(\tau)} \neq 0 \;\Leftrightarrow\; \lim_{\tau\to 0} \frac{\tau}{L(\tau)} = \frac{r^{iv}(0)}{2} \;\Leftrightarrow\; r^{iv}(0) < +\infty$$

or

$$\lim_{\tau\to 0} \frac{\tau}{L(\tau)} = 0 \;\Leftrightarrow\; r^{iv}(0) = +\infty.$$

Proof.

(i) Let us remark that

$$L(\tau) = \frac{r''(\tau) - r''(0)}{\tau} = \int_{-\infty}^{\infty} \frac{1 - e^{i\tau\lambda}}{\tau}\lambda^2 d\mu(\lambda) = 2\int_0^{\infty} \frac{1-\cos(\tau\lambda)}{\tau}\lambda^2 d\mu(\lambda).$$

Under the hypothesis $\mu_4 = +\infty$, Fatou lemma implies

$$\liminf_{\tau\to 0} \frac{L(\tau)}{\tau} \geq \int_0^{\infty} \liminf_{\tau\to 0} \frac{1-\cos(\tau\lambda)}{\frac{\tau^2\lambda}{2}}\lambda^4 d\mu(\lambda) = \int_0^{\infty} \lambda^4 d\mu(\lambda) = +\infty,$$

and the result follows.

(ii) If $\mu_4 < \infty$, the dominated convergence theorem implies

$$\lim_{\tau\to 0} \frac{L(\tau)}{\tau} = \int_0^{\infty} \lim_{\tau\to 0} \frac{(1-\cos(\tau\lambda))}{\tau^2\lambda^2/2}\lambda^4 d\mu(\lambda) = \int_0^{\infty} \lambda^4 d\mu(\lambda) = \frac{r^{(iv)}(0)}{2} > 0. \qquad \square$$

Lemma 2 *For τ belonging to a neighborhood of* 0*,*

(i) $\left|\dfrac{r'(\tau)}{\sigma(\tau)}\right|$ *is bounded.*

(ii) $\rho(\tau) \leq 0$.

Proof.

(i) It is a direct consequence of the previous lemma, (i) and (ii).
Indeed, $\dfrac{\tau}{L(\tau)}$ having always a limit, we can use L'Hopital rule and then write $\dfrac{(r'(\tau))^2}{\sigma^2(\tau)} \sim \dfrac{(r'(\tau))^2 2(1-r(\tau))}{-2r''(0)(1-r(\tau)) - (r'(\tau))^2} \sim \dfrac{\tau}{L(\tau)}$. The result follows from this last equivalence and the non vanishing property of $\sigma^2(\tau)$ for $\tau > 0$.

(ii) The sign of $\rho(\tau)$ is determined by the sign of $S(\tau) := -r''(\tau)\left(1 - r^2(\tau)\right) - r'^{\,2}(\tau)r(\tau)$.
But $S(\tau) \leq \; - \; \left[2r''(\tau)\left(1 - r(\tau)\right) + r'^{\,2}(\tau)r(\tau)\right]$ and
$\left[2r''(\tau)\left(1-r(\tau)\right) + r'^{\,2}(\tau)r(\tau)\right] \;\sim\; (-r''(0))\left(\tau^2\theta''(\tau) + 2\left(\theta(\tau) - \tau\theta'(\tau)\right) + \dfrac{(r''(0))^2}{2}\tau^4\right)$.
Let consider two cases depending on the existence of the fourth spectral moment.
- If $\mu_4 = \infty$, then

$$\begin{aligned} \tau^2\theta''(\tau) + \frac{(r''(0))^2}{2}\tau^4 &= \tau^2\theta''(\tau)\left(1 + \frac{(r''(0))^2}{2}\frac{\tau}{L(\tau)}\right) \\ &\sim \tau^2\theta''(\tau), \quad \text{because of } (i) \text{ of Lemma 1.} \end{aligned}$$

Hence $\left[2r''(\tau)\left(1-r(\tau)\right)+r'^{\,2}(\tau)r(\tau)\right] \ \sim\ \tau^2\theta''(\tau)-2\left(\tau\theta'(\tau)-\theta(\tau)\right)$.
We can show that this last quantity is positive, when writing

$$\begin{aligned}
\tau^2\theta''(\tau)+2\left(\theta(\tau)-\tau\theta'(\tau)\right) &= 2\left(\int_0^\tau\int_0^u \theta''(\tau)dvdu+\int_0^\tau\int_0^u \theta''(v)dvdu-\int_0^\tau\int_0^\tau \theta''(v)dvdu\right)\\
&= 2\left(\int_0^\tau\int_0^u \theta''(\tau)dvdu-\int_0^\tau\int_u^\tau \theta''(v)dvdu\right)\\
&= 2\int_0^\tau\int_u^\tau \left(\theta''(\tau)-\theta''(v)\right)dvdu\\
&= 2\int_0^\tau\int_u^\tau \left(r''(\tau)-r''(v)\right)dvdu,
\end{aligned}$$

and by noticing that the function $(-r'')$ is decreasing in a neighborhood of 0.
Therefore we have that $S(\tau)\le 0$, and so is $\rho(\tau)$.
- Suppose now that $\mu_4<\infty$. We have

$$\tau^2\theta''(\tau)+2\left(\theta(\tau)-\tau\theta'(\tau)\right)+\frac{(r''(0))^2}{2}\tau^4=\tau^4\left(\frac{L(\tau)}{\tau}-2\frac{\theta(\tau)-\tau\theta'(\tau)}{\tau^4}+\frac{(r''(0))^2}{2}\right)$$

and, since $\lim_{\tau\to 0}\dfrac{\tau\theta'(\tau)-\theta(\tau)}{\tau^4}=\dfrac{r^{iv}(0)}{8}$, then

$$\lim_{\tau\to 0}\tau^4\left(\frac{L(\tau)}{\tau}-2\frac{\theta(\tau)-\tau\theta'(\tau)}{\tau^4}+\frac{(r''(0))^2}{2}\right)=\frac{1}{2}\left(\frac{r^{iv}(0)}{2}+(r''(0))^2\right)>0,$$

from which we deduce that $\tau^2\theta''(\tau)+2\left(\theta(\tau)-\tau\theta'(\tau)\right)+\dfrac{(r''(0))^2}{2}\tau^4\sim\tau^4$.
Therefore $S(\tau)\le 0$ for all τ belonging to a neighborhood of 0. □

Lemma 3 *For τ belonging to a neighborhood of* 0*,*

(i) $\dfrac{\sigma^2(\tau)}{\tau}\ \le\ L(\tau)\ \le\ (2+C)\dfrac{\sigma^2(\tau)}{\tau}$*, with* $C\ge 0$*;*

(ii) For $\delta>0$*,*

$$\int_0^\delta \frac{\sigma^2(\tau)}{\sqrt{1-r^2(\tau)}}d\tau<\infty\ \Leftrightarrow\ \int_0^\delta L(\tau)d\tau<\infty\ (\textit{Geman condition}).$$

Proof.

(i) We can write

$$\rho(\tau)=1-\frac{r''(\tau)-r''(0)}{\sigma^2(\tau)}+\frac{r'^2(\tau)}{(1+r(\tau))\sigma^2(\tau)},$$

and since $-1\le\rho(\tau)\le 0$, we get

$$1\le 1+\frac{r'^2(\tau)}{(1+r(\tau))\sigma^2(\tau)}\le\frac{r''(\tau)-r''(0)}{\sigma^2(\tau)}\le 2+\frac{r'^2(\tau)}{(1+r(\tau))\sigma^2(\tau)}\le 2+C,$$

by applying (i) of Lemma 2. The definition of θ'' allows then to conclude.

(ii) This result can be easily deduced from the result (i), since $\sqrt{1-r^2(\tau)} \sim \tau$.
It is also interesting to notice that we can get this result by a direct computation, since $\dfrac{\sigma^2(\tau)}{\sqrt{1-r^2(\tau)}} \sim \left(\dfrac{\theta'(\tau)}{\tau^2} - \dfrac{\theta(\tau)}{\tau^3}\right)$ and, by integrating by parts,
$\displaystyle\int_0^\delta \left(\frac{\theta'(\tau)}{\tau^2} - \frac{\theta(\tau)}{\tau^3}\right) d\tau = \left[-\frac{\theta(\tau)}{2\tau^2} - \frac{\theta'(\tau)}{\tau}\right]_0^\delta + \frac{1}{2}\int_0^\delta \frac{\theta''(\tau)}{\tau} d\tau.$ □

To work on the necessary condition of the Theorem, the main tool will be the expansion into Hermite polynomials.
Recall that the Hermite polynomials $(H_n)_{n\geq 0}$ defined by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},$$

constitutes a complete orthogonal system in the Hilbert space $L^2(\mathbf{R}, \varphi(u)du)$.
In what follows, we will need the Hermite expansion of the function $|\cdot - m|$, m being some constant. We have

$$|x-m| = \sum_{l=0}^{\infty} a_l(m) H_l(x)$$

where

$$\begin{aligned} a_0(m) &= \mathrm{E}\,|Z-m|, \quad Z \text{ being a standard Gaussian r.v.,} \\ &= m\,[2\Phi(m)-1] + \sqrt{\frac{2}{\pi}}\, e^{-\frac{m^2}{2}} = \sqrt{\frac{2}{\pi}}\left[1 + \int_0^m \int_0^u e^{-\frac{v^2}{2}} dv du\right], \end{aligned} \tag{11}$$

$$a_1(m) = (1-2\Phi(m)) = -\sqrt{\frac{2}{\pi}} \int_0^m e^{-\frac{u^2}{2}} du = -m\sqrt{\frac{2}{\pi}} \int_0^1 e^{-\frac{m^2y^2}{2}} dy, \tag{12}$$

and

$$a_l(m) = \sqrt{\frac{2}{\pi}} \frac{1}{l!} H_{l-2}(m) e^{-\frac{m^2}{2}}, \qquad l \geq 2. \tag{13}$$

Note that $a_1(m) = 0$ if and only if $m = 0$, that

$$m a_1(m) = m[\Phi(-m) - \Phi(m)] < 0, \ \forall m, \tag{14}$$

and that

$$a_0(m) = -m a_1(m) + \sqrt{\frac{2}{\pi}} e^{-\frac{m^2}{2}} \geq \max\left(-m a_1(m), \sqrt{\frac{2}{\pi}} e^{-\frac{m^2}{2}}\right) \ > \ 0, \tag{15}$$

since $|m a_1(m)|$ and $e^{-\frac{m^2}{2}}$ can not be or tend to 0 simultaneously.

We are going to consider different cases of interest.
C will denote a positive constant which may vary from equation to equation.

• Suppose $\dot{\psi}_s = 0$ and $\psi_s \equiv x$, $\forall s$.
By using the regression (R), $I(t_1, t_1+\tau)$ can be rewritten as

$$I := p_\tau(x,x)\,\mathrm{E}\left|\left(\zeta + \frac{r'(\tau)}{1+r(\tau)}x\right)\left(\zeta^* - \frac{r'(\tau)}{1+r(\tau)}x\right)\right|.$$

We can consider two subcases, one well known when $x=0$ and the other when $x \neq 0$.
i) *Case when* $x = 0$.
In this particular situation, we have

$$I = p_\tau(0,0)\,\mathrm{E}\,|\zeta\zeta^*| \quad \text{and} \quad M_2 = 2\int_0^t (t-\tau)p_\tau(0,0)\sigma^2(\tau)\,\mathrm{E}\left|\frac{\zeta\zeta^*}{\sigma^2(\tau)}\right| d\tau.$$

We obtain, by using Mehler's formula,

$$M_2 = \frac{1}{\pi}\int_0^t (t-\tau)\frac{\sigma^2(\tau)}{\sqrt{1-r^2(\tau)}}\sum_{q=0}^{\infty} a_{2k}^2 (2k)!(\rho(\tau))^{2k} d\tau,$$

where the coefficients a_{2k} correspond to the Hermite expansion of the function $A(.) = |.|$ given in (11)-(13). Hence the following inequalities can be deduced:

$$\frac{a_0}{\pi}\int_0^t (t-\tau)\frac{\sigma^2(\tau)}{\sqrt{1-r^2(\tau)}}d\tau \leq M_2 \leq \frac{||A||^2_{L^2(\varphi)}}{\pi}\int_0^t (t-\tau)\frac{\sigma^2(\tau)}{\sqrt{1-r^2(\tau)}}d\tau,$$

which implies

$$M_2 < \infty \Leftrightarrow \int_0^t \frac{\sigma^2(\tau)}{\sqrt{1-r^2(\tau)}}d\tau < \infty. \tag{16}$$

The study of this last integral on $[0,t]$ reduces to the one on $[0,\delta]$, $\delta \in \mathcal{V}(0)$, because of the uniform continuity outside of a neighborhood $\mathcal{V}(0)$ of zero.
Combining (16) and (ii) of Lemma 3 allows to conclude that a necessary and sufficient condition to have $M_2 < \infty$ is that $L \in L^1[0,\delta]$. Thus we find back Geman's result ([7]).

ii) *Suppose now that* $x \neq 0$.
Then M_2^x can be written as

$$M_2^x = 2\int_0^t (t-\tau)p_\tau(x,x)\sigma^2(\tau)A(m,\rho,\tau)d\tau,$$

where $A(m,\rho,\tau) := \mathrm{E}\left|\left(\frac{\zeta}{\sigma(\tau)} + \frac{r'(\tau)}{(1+r(\tau))\sigma(\tau)}x\right)\left(\frac{\zeta^*}{\sigma(\tau)} - \frac{r'(\tau)}{(1+r(\tau))\sigma(\tau)}x\right)\right|$.
Note that

$$M_2^x \geq M_2^{x,\delta} := 2\int_0^\delta (t-\tau)p_\tau(x,x)\sigma^2(\tau)A(m,\rho,\tau)d\tau, \qquad \delta \in [0,\tau]. \tag{17}$$

Now, by using Mehler's formula, we have

$$A(m,\rho,\tau) = \sum_{k=0}^{\infty} a_k(m)a_k(-m)k!\rho^k(\tau),$$

where $a_k(m)$ are the Hermite coefficients of the function $|\cdot - m|$ given in (11)-(13) and

$$m = m(\tau) := \frac{r'(\tau)x}{(1+r(\tau))\sigma(\tau)},$$

$|m| = |m(\tau)|$ being bounded because of (i) of Lemma 2.

- Let us show that $M_2^x < \infty$ under the Geman condition.
Since by Cauchy-Schwarz inequality

$$|A(m,\rho,\tau)| \leq \sum_{k=0}^{\infty} |a_k(m)a_k(-m)|k! \leq \left(\mathrm{E}[(Y-m)^2]\,\mathrm{E}[(Y+m)^2]\right)^{1/2},$$

with Y standard normal r.v., it comes

$$M_2^x \leq I_2 := 2\int_0^t (t-\tau)p_\tau(x,x)\sigma^2(\tau)\left(a_0(m)a_0(-m) + 1 + m^2\right)d\tau.$$

Hence, m^2 being bounded, we obtain $I_2 \leq C\displaystyle\int_0^t (t-\tau)p_\tau(x,x)\sigma^2(\tau)d\tau$,
and we can prove in the same way as we did for the case $x=0$, that this integral is finite if $L \in L^1[0,\delta]$.

- Let us look now at the reverse implication.
Suppose that $M_2^x < \infty$, and so, via (17), that $M_2^{x,\delta} < \infty$.
Let us compute $A(m,\rho,\tau)$ and bound it below.
By using the parity of the Hermite polynomials, the sign of ρ given in (ii) of Lemma 2, and finally (15) and (14), we obtain

$$\begin{aligned}
A(m,\rho,\tau) &= a_0^2(m) + |\rho(\tau)|a_1^2(m) + \sum_{k=1}^{\infty} a_{2k}^2(m)(2k)!\rho^{2k}(\tau) + |\rho|\sum_{k=1}^{\infty} a_{2k+1}^2(m)(2k+1)!\rho^{2k}(\tau) \\
&\geq a_0^2(m) = \left(-ma_1(m) + \sqrt{\frac{2}{\pi}}e^{-\frac{m^2}{2}}\right)^2 \\
&\geq \frac{2}{\pi}e^{-m^2} \geq C, \quad \text{since } |m| < \infty.
\end{aligned}$$

Hence

$$\begin{aligned}
M_2^{x,\delta} &\geq C\int_0^\delta (t-\tau)p_\tau(x,x)\sigma^2(\tau)d\tau \\
&\geq C\int_0^\delta (t-\tau)\frac{\sigma^2(\tau)}{\sqrt{1-r^2(\tau)}}d\tau.
\end{aligned}$$

An application of Lemma 3, (ii), yields that $M_2^{x,\delta}<\infty$ implies the Geman condition.

• Suppose now the general case, i.e. $\dot\psi\neq 0$.
In this case, we consider that the function ψ satisfies

$$\dot\psi(t+\tau)=\dot\psi(t)+\gamma(t,\tau),\quad \text{for } \tau\in[0,\delta],\ 0\le\delta<1,$$

and that the modulus of continuity of $\dot\psi$ defined by $\tilde\gamma(\tau):=\sup_{u\in[0,t]}\sup_{|s|\le\tau}|\dot\psi(u+s)-\dot\psi(u)|$ verifies

$$\int_0^\delta \frac{\tilde\gamma(s)}{s}ds<\infty. \tag{18}$$

We have trivially $|\gamma(t,\tau)|\le\tilde\gamma(\tau)$, as well as $|\gamma(t,\varepsilon\tau)|\le\tilde\gamma(\tau)$, $\forall\ 0<\varepsilon<1$.

Remark: This smooth condition is satisfied by a large class of functions which includes in particular functions whose derivatives are Hölder.

Recall that M_2^ψ satisfies (4).
As for the previous cases, the study of M_2^ψ requires only to consider the case when τ belongs to a neighborhood of 0.

- Let us bound the term $I(t_1,t_1+\tau)$.
We have

$$\begin{aligned} I(t_1,t_1+\tau) &= p_\tau(\psi_{t_1},\psi_{t_1+\tau})\,\mathrm{E}\left(|\dot X_{t_1}-\dot\psi_{t_1}||\dot X_{t_1+\tau}-\dot\psi_{t_1+\tau}|\mid X_{t_1}=\psi_{t_1},X_{t_1+\tau}=\psi_{t_1+\tau}\right)\\ &= p_\tau(\psi_{t_1},\psi_{t_1+\tau})\,\mathrm{E}\left(|\dot X_{t_1}-\dot\psi_{t_1}||\dot X_{t_1+\tau}-\dot\psi_{t_1}-\gamma(t_1,\tau)|\mid X_{t_1}=\psi_{t_1},X_{t_1+\tau}=\psi_{t_1+\tau}\right)\\ &\le p_\tau(\psi_{t_1},\psi_{t_1+\tau})(E_1+E_2),\end{aligned}$$

with

$$E_1:=\mathrm{E}\left(|\dot X_{t_1}-\dot\psi_{t_1}||\dot X_{t_1+\tau}-\dot\psi_{t_1}|\mid X_{t_1}=\psi_{t_1},X_{t_1+\tau}=\psi_{t_1+\tau}\right)$$

and

$$E_2:=|\gamma(t_1,\tau)|\,\mathrm{E}\left(|\dot X_{t_1}-\dot\psi_{t_1}|\mid X_{t_1}=\psi_{t_1},X_{t_1+\tau}=\psi_{t_1+\tau}\right).$$

But, by using the regression (R), we have

$$E_1=\mathrm{E}\left|(\zeta+\alpha_1\psi_{t_1}+\alpha_2\psi_{t_1+\tau}-\dot\psi_{t_1})(\zeta^*-\beta_1\psi_{t_1}-\beta_2\psi_{t_1+\tau}-\dot\psi_{t_1})\right|=$$

$$\mathrm{E}\left|(\zeta+\alpha_1(\psi_{t_1}-\psi_{t_1+\tau})+(\alpha_1+\alpha_2)\psi_{t_1+\tau}-\dot\psi_{t_1})(\zeta^*-\beta_1(\psi_{t_1}-\psi_{t_1+\tau})-(\beta_1+\beta_2)\psi_{t_1+\tau}-\dot\psi_{t_1})\right|.$$

Note that, by using Taylor expansion, there exists $0<\varepsilon<1$ and $0<\varepsilon^*<1$ such that

$$\begin{aligned}&\zeta+\alpha_1(\psi_{t_1}-\psi_{t_1+\tau})+(\alpha_1+\alpha_2)\psi_{t_1+\tau}-\dot\psi_{t_1}=\\ &\zeta-\alpha_1\tau\left(\frac{\psi_{t_1+\tau}-\psi_{t_1}}{\tau}-\dot\psi_{t_1}\right)+(\alpha_1+\alpha_2)\psi_{t_1+\tau}-(1+\alpha_1\tau)\dot\psi_{t_1}=\\ &\zeta+(\alpha_1+\alpha_2)\psi_{t_1+\tau}-\alpha_1\tau\gamma(t_1,\varepsilon\tau)-(1+\alpha_1\tau)\dot\psi_{t_1},\end{aligned}$$

and, in the same way,

$$\zeta^* - \beta_1(\psi_{t_1} - \psi_{t_1+\tau}) - (\beta_1+\beta_2)\psi_{t_1+\tau} - \dot\psi_{t_1} = \zeta^* - (\alpha_1+\alpha_2)\psi_{t_1+\tau} + \beta_1\tau\gamma(t_1,\varepsilon^*\tau) - (1-\beta_1\tau)\,\dot\psi_{t_1}.$$

Hence

$$\begin{aligned}
E_1 \;\le\;& \mathrm{E}\,|(\zeta + (\alpha_1+\alpha_2)\psi_{t_1+\tau})\,(\zeta^* - (\alpha_1+\alpha_2)\psi_{t_1+\tau})| \\
&+ \left|\alpha_1\tau\gamma(t_1,\varepsilon\tau) + (1+\alpha_1\tau)\dot\psi_{t_1}\right| \mathrm{E}\,|\zeta^* - (\alpha_1+\alpha_2)\,\psi_{t_1+\tau}| \\
&+ \left|\beta_1\tau\gamma(t_1,\varepsilon^*\tau) - (1-\beta_1\tau)\,\dot\psi_{t_1}\right| \mathrm{E}\,|\zeta + (\alpha_1+\alpha_2)\psi_{t_1+\tau}| \\
&+ \left|\left(\alpha_1\tau\gamma(t_1,\varepsilon\tau) + (1+\alpha_1\tau)\dot\psi_{t_1}\right)\left(-\beta_1\tau\gamma(t_1,\varepsilon^*\tau) + (1-\beta_1\tau)\,\dot\psi_{t_1}\right)\right| \\
\le\;& \mathrm{E}\,|(\zeta + (\alpha_1+\alpha_2)\psi_{t_1+\tau})\,(\zeta^* - (\alpha_1+\alpha_2)\psi_{t_1+\tau})| \\
&+ C\left(\left|\alpha_1\tau\gamma(t_1,\varepsilon\tau) + (1+\alpha_1\tau)\dot\psi_{t_1}\right| + \left|\beta_1\tau\gamma(t_1,\varepsilon^*\tau) - (1-\beta_1\tau)\,\dot\psi_{t_1}\right|\right) \\
&+ \left|\left(\alpha_1\tau\gamma(t_1,\varepsilon\tau) + (1+\alpha_1\tau)\dot\psi_{t_1}\right)\left(\beta_1\tau\gamma(t_1,\varepsilon^*\tau) - (1-\beta_1\tau)\,\dot\psi_{t_1}\right)\right| \\
\le\;& \mathrm{E}\,|(\zeta + (\alpha_1+\alpha_2)\psi_{t_1+\tau})\,(\zeta^* - (\alpha_1+\alpha_2)\psi_{t_1+\tau})| + C\left\{|\alpha_1\tau|\tilde\gamma(\tau) + |(1+\alpha_1\tau)\dot\psi_{t_1}|\right. \\
&\left. + \left(|\alpha_1\tau|\tilde\gamma(\tau) + |(1+\alpha_1\tau)\dot\psi_{t_1}|\right)^2\right\},
\end{aligned}$$

because of (8) and of the definition of β_i $(i=1,2)$.
But, by using (7) and (i) of Lemma 3, we have for $\tau \in [0,\delta]$,

$$1 + \alpha_1\tau \;\sim\; \left(\sigma^2 - \frac{\theta}{\tau^2}\right) \le \sigma^2 \le C\;\tau L(\tau) \tag{19}$$

and

$$|\alpha_1\tau|\tilde\gamma(\tau) \;\sim\; \left|\left(-1 + \frac{\theta'(\tau)}{-r''(0)\tau}\right)\right|\tilde\gamma(\tau) \;\sim\; \tilde\gamma(\tau);$$

therefore

$$E_1 \le \mathrm{E}\,|(\zeta + (\alpha_1+\alpha_2)\psi_{t_1+\tau})\,(\zeta^* - (\alpha_1+\alpha_2)\psi_{t_1+\tau})| + C\left(|\tilde\gamma(\tau)| + |1+\alpha_1\tau|\right). \tag{20}$$

Moreover, by using Jensen conditional inequality and the fact that the conditional variance of a component of a Gaussian vector given the other components does not exceed the unconditional one, we have

$$E_2 \;\le\; \tilde\gamma(\tau)\left(\mathrm{E}\,|\dot X_{t_1+\tau} - \dot\psi_{t_1+\tau}|^2\right)^{1/2}. \tag{21}$$

So combining (20) and (21) provides

$$
\begin{aligned}
M_2^{\psi,\delta} &:= 2\int_0^t\int_0^\delta I(t_1,t_1+\tau)d\tau dt_1\\
&\le 2\int_0^t\int_0^\delta p_\tau(\psi_{t_1},\psi_{t_1+\tau})E_1(t_1,\tau)d\tau dt_1\\
&\qquad\qquad +2\int_0^t\int_0^\delta \left(\mathrm{E}\,|\dot X_{t_1+\tau}-\dot\psi_{t_1+\tau}|^2\right)^{1/2} p_\tau(\psi_{t_1},\psi_{t_1+\tau})\tilde\gamma(\tau)d\tau dt_1\\
&\le 2\int_0^t\int_0^\delta p_\tau(\psi_{t_1},\psi_{t_1+\tau})\,\mathrm{E}\,|(\zeta+(\alpha_1+\alpha_2)\psi_{t_1+\tau})\,(\zeta^*-(\alpha_1+\alpha_2)\psi_{t_1+\tau})|\,d\tau dt_1\\
&\quad + C\int_0^t\int_0^\delta p_\tau(\psi_{t_1},\psi_{t_1+\tau})\,(\tilde\gamma(\tau)+|1+\alpha_1\tau|)\,d\tau dt_1\\
&\quad + 2\int_0^t\int_0^\delta \left(\mathrm{E}\,|\dot X_{t_1+\tau}-\dot\psi_{t_1+\tau}|^2\right)^{1/2} p_\tau(\psi_{t_1},\psi_{t_1+\tau})\tilde\gamma(\tau)d\tau dt_1.
\end{aligned}
$$

Since $p_\tau(\psi_{t_1},\psi_{t_1+\tau})\le \dfrac{1}{2\pi\sqrt{1-r}} \;\sim\; \dfrac{1}{\tau}$, we obtain

$$
\begin{aligned}
M_2^{\psi,\delta} &= 2\int_0^t\int_0^\delta I(t_1,t_1+\tau)d\tau dt_1\\
&\le 2\int_0^t\int_0^\delta p_\tau(\psi_{t_1},\psi_{t_1+\tau})\,\mathrm{E}\,|(\zeta+(\alpha_1+\alpha_2)\psi_{t_1+\tau})\,(\zeta^*-(\alpha_1+\alpha_2)\psi_{t_1+\tau})|\,d\tau dt_1\\
&+\;C\int_0^t\int_0^\delta \left(1+\left(\mathrm{E}\,|\dot X_{t_1+\tau}-\dot\psi_{t_1+\tau}|^2\right)^{1/2}\right)\frac{\tilde\gamma(\tau)}{\tau}\,d\tau dt_1 + Ct\int_0^\delta \frac{|1+\alpha_1\tau|}{\tau}d\tau\\
&\le 2\int_0^t\int_0^\delta p_\tau(\psi_{t_1},\psi_{t_1+\tau})\,\mathrm{E}\,|(\zeta+(\alpha_1+\alpha_2)\psi_{t_1+\tau})\,(\zeta^*-(\alpha_1+\alpha_2)\psi_{t_1+\tau})|\,d\tau dt_1\\
&\quad +Ct\left(\int_0^\delta \frac{\tilde\gamma(\tau)}{\tau}\,d\tau+\int_0^\delta L(\tau)d\tau\right).
\end{aligned}
$$

The first integral on the RHS is finite under the Geman condition by applying the previous proof for a fixed level and the second one is finite under the condition (18) on $\tilde\gamma$, and the Geman condition.
Therefore we can conclude that $M_2^{\psi,\delta}<\infty$ and so that $M_2^{\psi}<\infty$ under the Geman condition and the condition (18).

- Let us prove now the reverse implication.
Suppose $M_2^{\psi}<\infty$, i.e. $M_2^{\psi,\delta}<\infty$.
By using the regression (R), we have

$$
\begin{aligned}
M_2^{\psi,\delta} &= 2\int_0^t\int_0^\delta p_\tau(\psi_{t_1},\psi_{t_1+\tau})E\left(|\dot X_{t_1}-\dot\psi_{t_1}||\dot X_{t_1+\tau}-\dot\psi_{t_1+\tau}|\mid X_{t_1}=\psi_{t_1},X_{t_1+\tau}=\psi_{t_1+\tau}\right)d\tau dt_1\\
&= 2\int_0^t\int_0^\delta p_\tau(\psi_{t_1},\psi_{t_1+\tau})\,\mathrm{E}\,|(\zeta+n_1)\,(\zeta^*+n_2)|\,d\tau dt_1,
\end{aligned}
$$

where $n_1 := n_1(t_1, \tau) = \alpha_1 \psi_{t_1} + \alpha_2 \psi_{t_1+\tau} - \dot{\psi}_{t_1}$
and $n_2 := n_2(t_1, \tau) = -\Big(\beta_1 \psi_{t_1} + \beta_2 \psi_{t_1+\tau} + \dot{\psi}_{t_1} + \gamma(t_1, \tau)\Big)$.
By using Taylor formula for ψ, we can write
$n_1 = m_1 + \alpha_2 \tau \gamma(t_1, \varepsilon\tau)$, with $m_1 := (\alpha_1 + \alpha_2)\psi_{t_1}(\tau) + (\alpha_2 \tau - 1)\dot{\psi}_{t_1}$, and
$n_2 := m_2 - \beta_2 \tau \gamma(t_1, \varepsilon^* \tau) - \gamma(t_1, \tau)$, with $m_2 = -[(\beta_1 + \beta_2)\psi_{t_1}(\tau) + (\beta_2 \tau + 1)\dot{\psi}_{t_1}]$.
The triangular inequality allows to write

$$\mathrm{E}\left|(\zeta + n_1)(\zeta^* + n_2)\right| \geq \mathrm{E}\left|(\zeta + m_1)(\zeta^* + m_2)\right| - I_1(\tau),$$

where

$$\begin{aligned} I_1(\tau) &= |\alpha_2 \tau \gamma(t_1, \varepsilon\tau)|\, \mathrm{E}\,|\zeta^* + m_2| + |\beta_2 \tau \gamma(t_1, \varepsilon^* \tau)|\, \mathrm{E}\,|\zeta + m_1| \\ &\quad + |\gamma(t_1, \tau)|\, \mathrm{E}\,|\zeta + m_1| + |\tau^2 \alpha_2 \beta_2 \gamma(t_1, \varepsilon\tau)\gamma(t_1, \varepsilon^* \tau)| + |\alpha_2 \tau \gamma(t_1, \varepsilon\tau)\gamma(t_1, \tau)| \\ &\leq C\{\tilde{\gamma}(\tau)\, \mathrm{E}\,|\zeta^* + m_2| + 2\tilde{\gamma}(\tau)\, \mathrm{E}\,|\zeta + m_1| + 2\tilde{\gamma}^2(\tau)\}, \end{aligned}$$

because of (8) and the definition of $\tilde{\gamma}$.
Moreover $\mathrm{E}\,|\zeta + m_1| < C$, $\mathrm{E}\,|\zeta^* + m_1| < C$, $\displaystyle\int_0^\delta \frac{\tilde{\gamma}(s)}{s} ds < \infty$.
Thus we obtain that there exists $\delta > 0$ such that

$$2 \int_0^t \int_0^\delta p_\tau(\psi_{t_1}, \psi_{t_1+\tau}) I_1(\tau) d\tau dt_1 < \epsilon,$$

from which we deduce that

$$M_2^{\psi,\delta} + \epsilon \geq\ 2 \int_0^t \int_0^\delta p_\tau(\psi_{t_1}, \psi_{t_1+\tau}) \sigma^2(\tau)\, \mathrm{E}\left| \left(\frac{\zeta + m_1}{\sigma(\tau)}\right) \left(\frac{\zeta^* + m_2}{\sigma(\tau)}\right) \right|.$$

But

$$\mathrm{E}\left| \left(\frac{\zeta + m_1}{\sigma(\tau)}\right) \left(\frac{\zeta + m_2}{\sigma(\tau)}\right) \right| = a_0(\tilde{m}_1) a_0(\tilde{m}_2) + a_1(\tilde{m}_1) a_1(\tilde{m}_2) \rho(\tau) + A_2(m_i, \rho, \tau), \qquad (22)$$

where $\tilde{m}_i := \dfrac{m_i}{\sigma(\tau)}$, for $i = 1, 2$, $A_2(m_i, \rho, \tau) = \displaystyle\sum_{k=2}^\infty a_k(\tilde{m}_1) a_k(\tilde{m}_2) k! \rho^k(\tau)$.
Let us study the term $A_2(m_i, \rho, \tau)$.
Since we can write $\dfrac{\rho^k(\tau)}{k!} = \dfrac{1}{(k-2)!} \displaystyle\int_0^{\rho(\tau)} \int_0^u v^{k-2} dv du$, we have

$$A_2(m_i, \rho, \tau) = \frac{2}{\pi} e^{-\frac{\tilde{m}_1^2 + \tilde{m}_2^2}{2}} \int_0^{\rho(\tau)} \int_0^u \sum_{k=0}^\infty \frac{1}{k!} H_k(\tilde{m}_1) H_k(\tilde{m}_2) v^k dv du.$$

By using the Hermite expansion for the two dimensional Gaussian density given by (see Berman ([2]) or Slud ([11], Lemma 5.1))

$$\sum_{k=0}^\infty \frac{v^k}{k!} H_k(\tilde{m}_1) H_k(\tilde{m}_2) = e^{\frac{\tilde{m}_1^2 + \tilde{m}_2^2}{2}} \frac{1}{\sqrt{1 - v^2}} e^{-\frac{\tilde{m}_1^2 + \tilde{m}_2^2 - 2v\tilde{m}_1\tilde{m}_2}{2(1-v^2)}},$$

we obtain that

$$A_2(m_i,\rho,\tau) = \frac{2}{\pi}\int_0^{\rho(\tau)}\int_0^{u} \frac{1}{\sqrt{1-v^2}} e^{-\frac{\tilde m_1^2+\tilde m_2^2-2v\tilde m_1\tilde m_2}{2(1-v^2)}} dvdu = \frac{2}{\pi}\int_0^{\rho(\tau)} \frac{\rho(\tau)-v}{\sqrt{1-v^2}} e^{-\frac{\tilde m_1^2+\tilde m_2^2-2v\tilde m_1\tilde m_2}{2(1-v^2)}} dv.$$

Hence

$$A_2(m_i,\rho,\tau) \;=\; \frac{2}{\pi}\int_0^{|\rho(\tau)|} \frac{|\rho(\tau)|-v}{\sqrt{1-v^2}} e^{-\frac{\tilde m_1^2+\tilde m_2^2+2v\tilde m_1\tilde m_2}{2(1-v^2)}} dv \;\geq\; 0. \qquad (23)$$

Let us now consider the term

$$A_{01} := a_0(\tilde m_1)a_0(\tilde m_2) + a_1(\tilde m_1)a_1(\tilde m_2)\rho(\tau) = a_0(\tilde m_1)a_0(\tilde m_2) - |\rho(\tau)|a_1(\tilde m_1)a_1(\tilde m_2).$$

Recall that $\tilde m_1 = \tilde m_1(\tau) = \dfrac{\alpha_1+\alpha_2}{\sigma(\tau)}\psi_{t_1} + \dfrac{\alpha_2\tau - 1}{\sigma(\tau)}\dot\psi_{t_1}$
and $\tilde m_2 = \tilde m_2(\tau) = \dfrac{\alpha_1+\alpha_2}{\sigma(\tau)}\psi_{t_1} + \dfrac{r(\tau)\alpha_2\tau - 1}{\sigma(\tau)}\dot\psi_{t_1} \sim \dfrac{\alpha_1+\alpha_2}{\sigma(\tau)}\psi_{t_1} + \dfrac{\alpha_2\tau - 1}{\sigma(\tau)}\dot\psi_{t_1}$,
since τ belongs to the neighborhood of 0.
But, as τ tends to 0, we have, by using (iii) of the lemma 3,

$$\left(\frac{\alpha_1+\alpha_2}{\sigma(\tau)}\right)^2 \;\sim\; \frac{\tau^2}{\sigma^2(\tau)} \;\sim\; \frac{\tau}{L(\tau)}$$

and

$$0 \leq \left(\frac{\alpha_2\tau - 1}{\sigma(\tau)}\right)^2 \;\sim\; \frac{\left(\frac{\sigma^2(\tau)}{2} - \frac{\theta}{\tau^2}\right)^2}{\sigma^2(\tau)} \;\leq \sigma^2(\tau) \;\to\; 0.$$

Therefore recalling Lemma 1 again,

- if $\dfrac{\tau}{L(\tau)} \;\to \dfrac{2}{r^{iv}(0)}$, as $\tau \to 0$,
  then $|\tilde m_i(\tau)| \;\sim\; \dfrac{\alpha_1+\alpha_2}{\sigma(\tau)}|\psi_{t_1}|$, for $i = 1, 2$, which implies that
  $\tilde m_1(\tau)\tilde m_2(\tau) \;\sim\; -\dfrac{(\alpha_1+\alpha_2)^2}{\sigma^2(\tau)}\psi_{t_1}^2$.
  Hence $\tilde m_1(\tau)\tilde m_2(\tau) \leq 0$ and so, because of (14), we have
  $-|\rho(\tau)|a_1(\tilde m_1)a_2(\tilde m_2) \geq 0$, which leads to $A_{01} \geq a_0(\tilde m_1)a_0(\tilde m_2)$.
  By using (15), it comes

  $$A_{01} \geq \max\left(|\tilde m_1 a_1(\tilde m_1)|, \sqrt{\frac{2}{\pi}}\; e^{-\frac{\tilde m_1^2}{2}}\right) \times \max\left(|\tilde m_2 a_1(\tilde m_2)|, \sqrt{\frac{2}{\pi}}\; e^{-\frac{\tilde m_2^2}{2}}\right) \geq C > 0; \;(24)$$

- if $\dfrac{\tau}{L(\tau)} \;\to\; 0$, as $\tau \to 0$,
  then $|\tilde m_i(\tau)| \;\to\; 0$ as $\tau \;\to\; 0$, for $i = 1, 2$, which implies that
  $a_1(\tilde m_i) \;\to\; 0$ as $\tau \;\to\; 0$, for $i = 1, 2$,

and $-|\rho(\tau)|a_1(\tilde{m}_i) \rightarrow 0$ as $\tau \rightarrow 0$, for $i = 1, 2$.

Hence by using once again (15), it comes

$$A_{01} \geq \frac{2}{\pi} e^{-\frac{\tilde{m}_1^2 + \tilde{m}_2^2}{2}} - |\rho(\tau)|a_1(\tilde{m}_1)a_1(\tilde{m}_2) \geq C > 0, \tag{25}$$

for τ sufficiently close to 0.

Combining (22), (23), (24) and (25) provides

$$M_2^{\psi,\delta} + \epsilon \geq C \int_0^t \int_0^\delta p_\tau(\psi_{t_1}, \psi_{t_1+\tau})\sigma^2(\tau)d\tau dt_1.$$

Since $p_\tau(\psi_{t_1}, \psi_{t_1+\tau}) \sim \dfrac{1}{\sqrt{1 - r^2(\tau)}}$, we obtain that

$$M_2^{\psi,\delta} + \epsilon \geq C \int_0^\delta \frac{\sigma^2(\tau)}{\sqrt{1 - r^2(\tau)}} d\tau,$$

which allows to conclude, by using (ii) of Lemma 3, that $M_2^{\psi,\delta} < \infty$, or $M_2^{\psi} < \infty$, implies the Geman condition.

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