Aggregation of random coefficient AR(1) process with infinite variance

(joint work with Donata Puplinskaitė)

Contents:

1. Aggregation: from "random" Markovian short memory towards "nonrandom" long memory

- Long memory and heavy tails are among the most widely discussed "stylized facts" of financial time series

- see e.g. Mikosch (2003): Modelling dependence and tails of financial time series

- Economic reasons for long memory (LM) are not clear

- LM can be partly explained by regime shifts and/or aggregation

- Granger (1980): (contemporaneous) aggregation of N random-coefficient AR(1) processes:

$$X_{it} = a_i X_{i,t-1} + \varepsilon_{it}, \qquad i = 1, 2, \cdots, N$$

- $\{\varepsilon_{it}, t \in \mathbb{Z}\}$: standardized i.i.d.; $a_i \in (0,1)$: *random*, independent of $\{\varepsilon_{it}\}$

- $\{X_{it}\}$, $i = 1, \cdots, N$: "micro-agents"; random parameter a: heterogeneity of micro-agents

- *idiosyncratic innovations*: $\{\varepsilon_{it}\}$, $i = 1, 2, \cdots$: independent (and identically distributed) $\Longrightarrow$ $\{X_{it}\}$, $i = 1, 2, \cdots$ independent (and identically distributed)

- *common innovations*: $\{\varepsilon_{it}\} \equiv \{\varepsilon_t\}$, $i = 1, 2, \cdots \Longrightarrow \{X_{it}\}$, $i = 1, 2, \cdots$ mutually dependent

• *Aggregated process*:

$$N^{-1/2}\sum_{i=1}^{N} X_{it} \ \xrightarrow{\text{fdd}}\ \bar{X}_t \qquad \text{(idiosyncratic scheme)}$$

$$N^{-1}\sum_{i=1}^{N} X_{it} \ \xrightarrow{\text{fdd}}\ \bar{X}_t \qquad \text{(common scheme)}$$

• spectral density of $\{\bar{X}_t\}$ (idiosyncratic scheme):

$$f_{\bar{X}}(z) = (2\pi)^{-1}\mathrm{E}\Big[\frac{1}{|1-a\mathrm{e}^{\mathrm{i}z}|^2}\Big] \ = \ (2\pi)^{-1}\mathrm{E}\Big[\frac{1}{(1-a)^2+4a\sin^2(z/2)}\Big], \qquad z\in[-\pi,\pi]$$

• If the (mixing) distribution $\Phi(\mathrm{d}a) := \mathrm{P}(a\in \mathrm{d}a)$ has a regularly varying at $a=1$ (= the unit root) probability density:

$$\phi(a) \ \sim \ \phi_1(1-a)^b, \qquad a\uparrow 1 \qquad (\exists\ 0<b<1,\ \phi_1>0), \tag{1}$$

then $f_{\bar{X}}(z)$ is *unbounded at* $z=0$ *and has a power behavior with exponent* $b-1\in(-1,0)$ *as* $z\to 0$:

$$\begin{aligned} f_{\bar{X}}(z) \ &\sim \ \frac{\phi_1}{2\pi}\int_0^1 \frac{(1-a)^b\mathrm{d}a}{(1-a)^2+4a\sin^2(z/2)} \\ &\sim \ \frac{\phi_1}{2\pi}\int_0^1 \frac{v^b\,\mathrm{d}v}{v^2+4\sin^2(z/2)} \\ &\sim \ \frac{\phi_1}{2\pi(2\sin(z/2))^{1-b}}\int_0^{1/2\sin(z/2)} \frac{w^b\mathrm{d}w}{w^2+1} \\ &\sim \ \frac{C}{z^{1-b}}, \qquad\qquad C := \tfrac{\phi_1}{2\pi}\textstyle\int_0^\infty \tfrac{w^b\mathrm{d}w}{w^2+1} \end{aligned}$$

- The above behavior of spectral density is characteristic to LM

- By the classical CLT, $\{\bar{X}_t\}$ is Gaussian

- Granger (1980): Beta-distributed a; Gonçalves and Gourieroux (1988), Zaffaroni (2004)

- particular mixing density $\phi(a) \propto a^{d-1}(1+a)(1-a)^{1-2d}$ (corresponding to $b = 1-2d$) leads to FARIMA$(0, d, 0)$ process $\{\bar{X}_t\}$ (Celov et al., 2007)

- Oppenheim and Viano (2001): aggregation of AR(p), $p \geq 1$ and seasonal LM

- singular mixing density (1) with $b < 0$: *nonstationary LM* (Zaffaroni, 2004)

- Common innovations: similar results

- Related problems:

\- *disaggregation*: statistical estimation of the mixing density from observed sample $\bar{X}_1, \cdots, \bar{X}_n$ (Celov et al., 2007)

- *aggregation of conditionally heteroskedastic models*: Ding and Granger (1996), Zaffaroni (2007a, 2007b), Giraitis et al. (2010)

Giraitis et al. (2010): aggregation of random-coefficient GLARCH(1,1) with common standardized innovations $\{\varepsilon_t\}$:

$$X_{it} \;=\; \varepsilon_t V_{it}, \qquad V_{it} \;=\; (1-a_i) + a_i V_{i,t-1} + c a_i \sqrt{1-a_i^2} X_{i,t-1}, \qquad i = 1, \cdots, N$$

$0 < c < 1$: parameter; $a_i \in (0,1)$: i.i.d. random coefficients, $a_i \sim$Beta(p,q), $p,q > 0$.

Then

$$N^{-1} \sum_{i=1}^{N} X_{it} \;\xrightarrow{\text{fdd}}\; \bar{X}_t,$$

where $\{\bar{X}_t = \varepsilon_t \bar{V}_t\}$ is a stochastic volatility process with LM (for $0 < q < 1/2$)

- For $0 < c$ small enough and $0 < q < 1/2$

$$\left.\begin{array}{c} n^{q-1} \sum_{s=1}^{[nt]} (\bar{V}_s^2 - \mathrm{E}\bar{V}_s^2) \\ n^{q-1} \sum_{s=1}^{[nt]} (\bar{X}_s^2 - \mathrm{E}\bar{X}_s^2) \end{array}\right\} \;\xrightarrow{\mathrm{D}[0,1]}\; 2Y_G(t) \qquad (n \to \infty), \tag{2}$$

where

$$Y_G(t) \;:=\; \int_{-\infty}^{t} \Big\{ \int_{0\vee s}^{t} G\Big(\frac{W(\tau) - W(s)}{\sqrt{\tau - s}}\Big) \frac{\mathrm{d}\tau}{(\tau - s)^{q+(1/2)}} \Big\} \mathrm{d}W(s), \quad t \geq 0$$

- $\{Y_G(t)\}$ is a stationary increment self-similar process with index $H := 1 - q \in (1/2, 1)$

• $\{Y_G(t)\}$ is well-defined for any $G : \mathbb{R} \to \mathbb{R}$ with $\mathrm{E}G^2(Z) < \infty$, $Z \sim N(0,1)$ and any $0 < q < 1/2$, as a backward Itô integral

• For $G \equiv const$, $\{Y_G(t)\}$ is a fractional Brownian motion

• In (2), G is a special function expressed via the degenerated hypergeometric function

- *aggregation of autoregressive random fields*: Lavancier (2005), Azomahou (2009)

• Conclusion: aggregation of simple dynamic AR(1) equations can lead to well-studied Gaussian or linear fractionally integrated I(d) models with long memory

2. Aggregation of infinite variance AR(1): common innovations

Puplinskaitė & S. (2009):

$$X_{it} = a_i X_{i,t-1} + \varepsilon_t, \qquad i = 1, 2, \cdots, N \tag{3}$$

$\{\varepsilon_t, t \in \mathbb{Z}\}$: i.i.d. in the domain of (normal) attraction of $\alpha-$stable law, $1 < \alpha \le 2$; $\mathrm{E}\varepsilon_t = 0$

$a_i \in (0,1)$: i.i.d., independent of $\{\varepsilon_t\}$; $\mathrm{E}[\frac{1}{(1-a^p)^{1/p}}] < \infty$ $(\exists p < \alpha)$

$\mathcal{A} := \sigma\{a_i, i = 1, 2, \cdots\}$

Then there exist stationary solution $\{X_{it}\}$ of (3) given by $X_{it} = \sum_{j=0}^{\infty} a_i^j \varepsilon_{t-j}$ and

$$N^{-1} \sum_{i=1}^{N} X_{it} \;=\; \sum_{j=0}^{\infty} \Big\{ N^{-1} \sum_{i=1}^{N} a_i^j \Big\} \varepsilon_{t-j} \;\stackrel{L^p(\mathcal{A})}{\longrightarrow}\; \bar{X}_t := \sum_{j=0}^{\infty} \bar{a}_j \varepsilon_{t-j}$$

where $\bar{a}_j := \mathrm{E}a^j$.

- Assume the mixing density satisfies

$$\phi(a) \;\sim\; \phi_1 (1-a)^{-d}, \qquad a \uparrow 1 \qquad (\exists\, d < 1,\ \phi_1 > 0) \tag{4}$$

Then the $\bar{a}_j$'s decay as j^{d-1} as $j \to \infty$ ($\sum_{j=0}^{\infty} |\bar{a}_j| = \infty$ when $0 < d < 1$):

$$\begin{aligned} \bar{a}_j \;&\sim\; \phi_1 \int_0^1 a^j (1-a)^{-d} \mathrm{d}a \\ &=\; \phi_1 \int_0^1 (1-v)^j v^{-d} \mathrm{d}v \\ &\sim\; \phi_1 \int_0^1 \mathrm{e}^{-vj} v^{-d} \mathrm{d}v \\ &\sim\; C j^{d-1}, \qquad C := \phi_1 \textstyle\int_0^\infty \mathrm{e}^{-w} w^{-d} \mathrm{d}w \end{aligned}$$

• The case of Beta-distributed $a \sim \mathrm{Beta}(p, 1-d)$ leads to FARIMA$(0, d, 0)$ process:

$$\bar{a}_j \;=\; \mathrm{E}a^j \;=\; \frac{1}{B(d, 1-d)} \int_0^1 a^j a^d (1-a)^{-d} \mathrm{d}a \;=\; \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)}$$

• Under (4), $\{\bar{X}_t\}$ is a well-defined MA process in i.i.d. innovations $\{\varepsilon_t\} \in D(\alpha)$ iff

$$0 \;<\; d \;<\; 1 - (1/\alpha). \tag{5}$$

• LM properties of such moving averages with regularly decaying coefficients are well-studied: Astrauskas (1983), Maejima (1983), Astrauskas et al. (1991), Avram and Taqqu (1992), Samorodnitsky and Taqqu (1994)

• Under (4), (5), partial sums of $\{\bar{X}_t\}$, normalized by $n^{1/\alpha+d}$, tend to $\alpha-$stable fractional motion, which is self-similar with index $H = (1/\alpha) + d \in (1/\alpha, 1)$

• $1 - (1/\alpha) < d < 1 \;\Rightarrow$ nonstationary aggregated process $\{\bar{X}_t\}$

• $\alpha = 2$: Zaffaroni (2004)

3. Aggregation of infinite variance AR(1): idiosyncratic innovations

AR(1) equation:

$$X_t = aX_{t-1} + \varepsilon_t, \qquad t \in \mathbb{Z} \tag{6}$$

$\{\varepsilon_t, t \in \mathbb{Z}\}$: i.i.d. in the domain of (normal) attraction of $\alpha-$stable r.v. $Z,\ 1 < \alpha \le 2;\ \mathrm{E}\varepsilon_t = 0$

$a \in (0,1)$: random and independent of $\{\varepsilon_t\}$;

$$\mathrm{E}[\frac{1}{1-a}] < \infty \tag{7}$$

$\{X_{it}\},\ i = 1, 2, \cdots$: independent copies of $\{X_t\}$ in (6)

Then

$$N^{-1/\alpha} \sum_{i=1}^{N} X_{it} \ \xrightarrow{\text{fdd}}\ \bar{X}_t \ := \ \sum_{s \le t} \int_0^1 a^{t-s} M_s(\mathrm{d}a), \tag{8}$$

where $\{M_s, s \in \mathbb{Z}\}$ are i.i.d. copies of an $\alpha-$stable random measure on $(0,1)$ with control measure $\Phi(\mathrm{d}a) := \mathrm{P}(a \in \mathrm{d}a)$ (the mixing distribution)

The characteristic functional of M_s:

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\sum_{i=1}^m \theta_i M_s(A_i)} \ = \ \exp\{-\sum_{i=1}^m |\theta_i|^\alpha \omega(\theta_i)\Phi(A_i)\} \ = \ \exp\{-\sum_{i=1}^m |\theta_i|^\alpha \omega(\theta_i)\mathrm{P}(a \in A_i)\},$$

$A_i \subset (0,1)$: any disjoint Borel sets, $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta Z} = \mathrm{e}^{-|\theta|^\alpha \omega(\theta)},\ \theta \in \mathbb{R}$

• If $\alpha = 2$, then (8) is Gaussian and

$$\mathrm{cov}(\bar{X}_0, \bar{X}_t) \;=\; \sum_{s \le 0} \int_0^1 a^{t-s} a^{-s} \Phi(\mathrm{d}a) \;=\; \mathrm{E} \frac{a^t}{1 - a^2} \;=\; \mathrm{cov}(X_0, X_t)$$

• (8): particular case of *mixed stable moving averages* (S., Rosinski, Mandrekar, Cambanis (1993))

• (8) is different from usual $\alpha-$stable moving average except when Φ is concentrated at a single point

• different mixing distributions Φ lead to different processes $\{\bar{X}_t\}$

• condition (7) is precise: if (7) is not satisfied, the limit aggregated process can be degenerated and $\alpha'-$stable, for $\alpha' < \alpha$

• $\{\varepsilon_s := M_s(0,1), s \in \mathbb{Z}\}$: i.i.d. sequence of $\alpha-$stable r.v.'s. Then

$$\mathrm{E}\Big[\sum_{j=0}^{\infty} \int_0^1 a^j M_{t-j}(\mathrm{d}a) \Big| \varepsilon_s, s \in \mathbb{Z} \Big] \;=\; \sum_{j=0}^{\infty} \mathrm{E}[a^j] \varepsilon_{t-j}. \tag{9}$$

(9) establishes a link between the aggregated processes in the idiosyncratic and common innovations schemes

• (9) follows from general interpolation formula (S., 1979)

4. Mixed stable moving averages and SαS stationary processes

A *mixed SαS moving average* $(0 < \alpha < 2)$ is a stationary process

$$Y(t) \;=\; \sum_{s\in\mathbb{Z}} \int_W g(v, t-s) M_s(\mathrm{d}v), \qquad t \in \mathbb{Z} \tag{10}$$

where:

W is a measurable space with $\sigma-$finite measure μ;

$g \in L_\alpha(W \times \mathbb{Z})$;

$\{M_s, s \in \mathbb{Z}\}$ are i.i.d. copies of a SαS random measure on W with control measure μ.

- mixed SαS moving averages generalize usual SαS moving averages and their sums

- finite-dimensional distributions of mixed SαS moving averages are SαS

- mixed SαS moving averages are ergodic and mixing

- The triplet (W, g, μ) determines the distribution of $\{Y(t)\}$ uniquely. The interesting question is to find conditions on (W_1, g_1, μ_1) and (W_2, g_2, μ_2) such that the distributions of the corresponding mixed SαS moving averages coincide: $\{Y_1(t)\} \stackrel{\text{fdd}}{=} \{Y_2(t)\}$

• S. et al. (1993): there is a 1-1 correspondence between the distribution of a mixed SαS moving average and a certain measure π on the unit sphere of the factor space $L_\alpha(\mathbb{Z})/\sim$ (with the equivalence up to sign and shift) (π is defined from (W, g, μ))

• mixed SαS moving averages play important role in the general theory of stationary SαS processes (Rosinski, 1995):

Any SαS stationary process $\{Y(t), t \in \mathbb{Z}\}$ $(0 < \alpha < 2)$ admits a stochastic integral representation

$$Y(t) = \int_E f_t(x) M(\mathrm{d}x),$$

where:

M is a SαS random measure on a measurable space E with $\sigma-$finite control measure μ;

$$f_t(x) \;=\; a_t(x)\Big[\frac{d\mu \circ \varphi_t}{d\mu}(x)\Big]^{1/\alpha}(f_0 \circ \varphi_t)(x);$$

where

$f_0 \in L_\alpha(E, \mu)$;

$\{\varphi_t\}$ is a nonsingular flow (a flow is a family of mappings $\varphi_t : E \to E$ such that $\varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}$ and φ_0 is identity);

$\{a_t\}$ is a cocycle (a family of mappings $a_t : E \to \{-1, 1\}$ such that $a_{t_1+t_2} = a_{t_2}(a_{t_1} \circ \varphi_{t_2})$).

Using Hopf's decomposition (of a given flow into conservative and dissipative parts), Rosinski (1995) showed that any SαS stationary process Y can be written as a sum of two independent SαS stationary processes Y_C ("conservative") and Y_D ("dissipative"): $Y = Y_C + Y_D$.

Furthermore, any "dissipative" SαS stationary process is a mixed SαS moving average (Rosinski, 1995)

5. Long memory properties of the aggregated process

The aggregated $\alpha-$stable process (mixed $\alpha-$stable moving average):

$$\bar{X}_t \; := \; \sum_{s \le t} \int_0^1 a^{t-s} M_s(\mathrm{d}a), \tag{11}$$

where

$\{M_s\}$ are i.i.d. $\alpha-$stable random measures on $(0,1)$ with control measure $\Phi(\mathrm{d}a) = \mathrm{P}(a \in \mathrm{d}a)$

Assume that Φ has a probability density ϕ regularly decaying at $a = 1$ with exponent $b > 0$:

$$\phi(a) \;\sim\; \phi_1 (1-a)^b, \qquad a \uparrow 1 \qquad (\exists\, \phi_1 > 0,\; b > 0) \tag{12}$$

We expect that for some values of the parameters α and b, the process in (11) will exhibit long memory, in a certain sense

5.1 Partial sums and self-similarity

DEFINITION 1 (Cox, 1984) A strictly stationary time series $\{Y_t\}$ is said to have *distributional long memory* or *distributional short memory* if there exist some constants $A_n \to \infty$ $(n \to \infty)$ and B_n, and a stochastic process $\{J(t), t \geq 0\} \not\equiv 0$ with *dependent increments* or, respectively, *independent increments* such that

$$A_n^{-1} \sum_{s=1}^{[nt]} (Y_s - B_n) \xrightarrow{\text{fdd}} J(t). \tag{13}$$

• Lamperti (1962): under mild additional assumptions, constants A_n in (13) grow as n^H with some $H > 0$ and the limit process $\{J(t), t \geq 0\}$ is *self-similar with index* H.

THEOREM 1

(i) Let $1 < \alpha < 2$ *and* $0 < b < \alpha - 1$. *Let* $H := 1 - b/\alpha$. *Then*

$$\frac{1}{n^H} \sum_{t=1}^{[n\tau]} \bar{X}_t \xrightarrow{\text{fdd}} Z(\tau),$$

where:

$$Z(\tau) := \int_{\mathbb{R}_+ \times \mathbb{R}} (f(x, \tau - s) - f(x, -s)) \nu(\mathrm{d}x, \mathrm{d}s), \tag{14}$$

$$f(x,t) := (1 - \mathrm{e}^{-xt}) \mathbf{1}(x > 0, t > 0),$$

ν : $\alpha-$*stable r. m. on* $(0, \infty) \times \mathbb{R}$ *with control measure* $\phi_1 x^{b-\alpha} \mathrm{d}x \mathrm{d}s$

(ii) Let $0 < \alpha < 2$ *and* $b > \max(\alpha - 1, 0)$*. Then*

$$\frac{1}{n^{1/\alpha}} \sum_{t=1}^{[n\tau]} \bar{X}_t \xrightarrow{\text{fdd}} \alpha - \textit{stable Lévy process}$$

- 3 different behaviors of the AR(1) aggregation scheme:

$1 < \alpha < 2, \; 0 < b < \alpha - 1$: distributional long memory

$0 < \alpha < 2, \; b > \max(\alpha - 1, 0)$: distributional short memory

$0 < \alpha < 2, \; -1 < b < 0$: degenerate: $\alpha(1+b)-$stable constant

- $\{Z(t)\}$ of (14) is $H-$sssi with $H = 1 - (b/\alpha) \in (1/\alpha, 1)$, has $\alpha-$stable distributions and a.s. continuous paths

- $\{Z(t)\}$ of (14) is different from fractional stable motion

- Related $\alpha-$stable $H-$sssi processes: S. et al. (1992), Cioczek-Georges et al. (1995), Cioczek-Georges and Mandelbrot (1995) ("fractal sums of pulses")

5.2 LRD(SAV) property

LRD(SAV) = Long Range Dependent (Sample Allen Variance)

Definition 2 (Heyde and Yang, 1997)

(i) A strictly stationary zero-mean process $\{Y_t\}$ is called *LRD(SAV)* if

$$\frac{\left(\sum_{t=1}^{n} Y_t\right)^2}{\sum_{t=1}^{n} Y_t^2}$$

tends to ∞ in probability as $n \to \infty$.

(ii) A strictly stationary zero-mean process $\{Y_t\}$ is called *SRD(SAV)* if the above ratio is bounded in probability.

- applies to finite and infinite variance processes

- requires finite mean (case $\alpha > 1$ only)

- If $\{Y_t\}$ are i.i.d., $Y_t \in D(\alpha)$, the ratio has a proper limit (Chistyakov and Götze, 2004)

- applicable to $\alpha-$stable moving averages and some other classes of LM processes (Leipus et al., 2006)

Theorem 2

(i) Let $1 < \alpha < 2$ *and* $0 < b < \alpha - 1$. *Then* $\{\bar{X}_t\}$ *is LRD(SAV).*

(ii) Let $1 < \alpha < 2$ *and* $b > \alpha - 1$. *Then* $\{\bar{X}_t\}$ *is SRD(SAV).*

- Thm 2 agrees with Thm 1

5.3 Codifference

Codifference (between Y_0 and Y_t) is defined (Samorodnitsky and Taqqu, 1994)

$$\begin{aligned}\mathrm{cod}(Y_0, Y_t) &:= \log \mathrm{E}\mathrm{e}^{\mathrm{i}(Y_t - Y_0)} - \log \mathrm{E}\mathrm{e}^{\mathrm{i}Y_t} - \log \mathrm{E}\mathrm{e}^{-\mathrm{i}Y_0} \\ &= \log\left(1 + \frac{\mathrm{cov}(\mathrm{e}^{\mathrm{i}Y_t}, \mathrm{e}^{\mathrm{i}Y_0})}{\mathrm{E}\mathrm{e}^{\mathrm{i}Y_t}\mathrm{E}\mathrm{e}^{-\mathrm{i}Y_0}}\right)\end{aligned}$$

- applies to finite and infinite variance processes (e.g., infinite variance moving averages)

- can be used to characterize the long memory of $\{Y_t\}$ and its intensity

- if $\{Y_t\}$ is a standardized Gaussian process then $\mathrm{cod}(Y_0, Y_t) = \frac{1}{2}\mathrm{cov}(Y_0, Y_t)$

- related measure of dependence: bivariate characteristic function (Astrauskas et al., 1991)

THEOREM 3 *Let* $0 < \alpha < 2$ *and* $0 < b < 1$. *Then*

$$\mathrm{cod}(\bar{X}_0, \bar{X}_t) \sim C\, t^{-b}, \qquad t \to \infty,$$

where

$$C := \phi_1 \alpha^{-1} \int_0^\infty [\omega(1)\mathrm{e}^{-y\alpha} + \overline{\omega(1)}(1 - (1 - \mathrm{e}^{-y})^\alpha)] y^{b-1} \mathrm{d}y.$$

• $\sum_{t=0}^{\infty} |\mathrm{cod}(\bar{X}_0, \bar{X}_t)| = \infty$ for $0 < b < 1$: "long memory"?

• For $0 < b < \alpha - 1$: Thm 3 agrees with the LM characterizations in Thms 1 and 2

• For $b > \max(0, \alpha - 1)$: Thm 3 disagrees with the LM characterizations in Thms 1 and 2 ?

• Contrary to distributional LM and LRD(SAV), the *codifference measures the dependence in* $\{\mathrm{e}^{\mathrm{i}\bar{X}_t}\}$ *rather than in* $\{\bar{X}_t\}$

• We conjecture that *for* $\max(0, \alpha - 1) < b < 1$, *the process* $\{\mathrm{e}^{\mathrm{i}\bar{X}_t}\}$ *has distributional short memory*, more precisely, that

$$n^{1/(1+b)} \sum_{i=1}^{[nt]} (\mathrm{e}^{\mathrm{i}\bar{X}_t} - \mathrm{E}\mathrm{e}^{\mathrm{i}\bar{X}_t}) \ \xrightarrow{\text{fdd}} \ (1+b)\text{-stable Lévy process}$$

with $1 + b > \alpha$ $(1 < \alpha < 2)$, despite the fact that the covariance of $\{\mathrm{e}^{\mathrm{i}\bar{X}_t}\}$ is not summable

• the above conjecture is based on corresponding results for bounded functionals of usual infinite variance moving averages (S. (2002), Honda (2010)):

Let $Y_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$, where $\{\varepsilon_j\}$ i.i.d., $\varepsilon_j \in D(\alpha)$, $c_j \sim j^{-\beta}$, and $1 < \beta < 2/\alpha$

Then

$$\mathrm{cod}(Y_0, Y_t) \sim const.t^{1-\alpha\beta}, \quad \text{hence} \quad \sum_{t=0}^{\infty} |\mathrm{cod}(Y_0, Y_t)| = \infty.$$

At the same time,

$$n^{1/(\alpha\beta)} \sum_{i=1}^{[nt]} (\mathrm{e}^{\mathrm{i}Y_t} - \mathrm{E}\mathrm{e}^{\mathrm{i}Y_t}) \xrightarrow{\text{fdd}} (\alpha\beta)\text{-stable Lévy process}$$

5.4 Ruin probability

Classical problem of insurance mathematics is the asymptotics of the *ruin probability*:

$$\psi(u) := \mathrm{P}\Big(\sup_{n\ge 1}(\sum_{t=1}^{n} Y_t - cn) > u\Big), \quad \text{as } u \to \infty, \tag{15}$$

where:

$\{Y_t\}$: random stationary (independent or dependent) sequence ("claims")

$c > 0$: constant deterministic premium rate ($c > \mathrm{E}Y_1 =: \mu$)

$u > 0$: the initial capital (of the insurance company)

$\Big\{\sup_{n\ge 1}(\sum_{t=1}^{n} Y_t - cn) > u\Big\}$: ruin occurs at some moment $n \ge 1$

- The "classical" result in the i.i.d. claims situation is that $\psi(u) = O(u^{-(\alpha-1)})$. If $\mathrm{P}(Y_1 > x) \sim c_\alpha x^{-\alpha}$ $(x \to \infty)$, $\alpha > 1$ then (Embrechts and Veraverbeke, 1982)

$$\psi(u) \;\sim\; \frac{c_\alpha}{(\alpha-1)(c-\mu)} u^{-(\alpha-1)}, \qquad u \to \infty. \tag{16}$$

- (16) "remains valid" for weakly dependent claims

• Mikosch and Samorodnitsky (2000): strongly dependent stationary SαS claims $\{Y_t\}$. See also Alparslan and Samorodnitsky (2007), Alparslan (2009)
• Mikosch and Samorodnitsky (2000) associate the 'classical' decay rate $\psi(u) = O(u^{-(\alpha-1)})$ with short-range dependence and the decay rate $\psi(u) = O(u^{-\nu})$ with exponent $\nu < \alpha - 1$ with long-range dependence of the claim sequence $\{Y_t\}$

• For stationary increments of linear $\alpha-$stable fractional motion with self-similarity parameter $H \in (1/\alpha, 1)$, Mikosch and Samorodnitsky (2000) obtained a decay rate $\psi(u) \sim (\text{constant})\, u^{-\alpha(1-H)}$ (different from (16)

• A natural problem is to extend the last result to the mixed $\alpha-$stable moving average process $\{\bar{X}_t\}$

THEOREM 4 *Let* $0 < b < \alpha - 1,\ 1 < \alpha < 2$ *and* $\{\bar{X}_t\}$ *be symmetric. Then*

$$\psi(u) \ \sim \ \frac{K(\alpha, b)}{c^{H\alpha}} u^{-\alpha(1-H)}, \qquad u \to \infty,$$

where:

$$K(\alpha, b) \ := \ (\phi_1/b) \int_0^\infty z^b g^\alpha(z) \mathrm{d}z + (\phi_1/\alpha) \int_0^\infty w^{b-1} g^\alpha(w) \mathrm{d}w,$$

$$H \ := \ 1 - \frac{b}{\alpha} \ \in \ (\frac{1}{\alpha}, 1) \qquad \textit{(the asymptotic self-similarity index)},$$

$$g(w) \ := \ \sup_{z>0} \frac{1 - \mathrm{e}^{-z}}{w + z} \quad \textit{(a continuous function with } g(0+) = 1,\ g(w) = O(1/w)\textit{)}$$

• The proof of Thm 4 uses the equivalence $\psi(u) \sim \psi_0(u)$ in Mikosch and Samorodnitsky (2000, Thm 2.5) to the heavy-tailed large deviation functional

$$\begin{aligned}\psi_0(u) \;:=\; & \frac{1}{2}\sum_{s\in\mathbb{Z}}\int_W \sup_{n\geq 1}\frac{\left(\sum_{t=1}^{n} f(v,t-s)\right)_+^{\alpha}}{(u+nc)^{\alpha}}\,\mu(\mathrm{d}v) \\ + \; & \frac{1}{2}\sum_{s\in\mathbb{Z}}\int_W \sup_{n\geq 1}\frac{\left(\sum_{t=1}^{n} f(v,t-s)\right)_-^{\alpha}}{(u+nc)^{\alpha}}\,\mu(\mathrm{d}v);\end{aligned}$$

where $Y_t = \sum_{s\in\mathbb{Z}}\int_W f(v,t-s)M_s(\mathrm{d}v)$ is a general mixed SαS moving average

• Samorodnitsky (2004) associated long memory with the rate of growth of maxima and partial maxima of a stationary $\alpha-$stable process. Theorem 4.1 of the above paper says that partial maxima of an SαS process generated by a dissipative flow always grow at the rate $n^{1/\alpha}$. Therefore, the rate of growth of the sequence of partial maxima is incapable of discriminating between long memory and short memory in the aggregate process in (8), since this process is a particular case of the class of mixed moving averages generated by dissipative flows.

6. Concluding remarks

- Aggregation of simple AR(1) process with heavy-tailed noise leads to a natural class of "AR(1)" mixed stable moving averages [= limits of sums of independent $\alpha-$stable AR(1) processes]. The dependence in the aggregated process is completely specified by the mixing distribution of the random AR(1) coefficient. If the mixing density has a power decay at the unit root $a = 1$, the aggregated $\alpha-$stable process displays long memory which can be characterized according to several definitions.

- There exists a notable "1-1 correspondence" between dependence properties of the aggregated process $\{\bar{X}_t\}$ with mixing density $\phi(a) \sim \phi_1(1-a)^b$, $0 < b < \alpha - 1$ and the corresponding properties of $\alpha-$stable moving average

$$Y_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \qquad c_j \sim c j^{-\beta}, \qquad \frac{1}{\alpha} < \beta < 1$$

The correspondence is effected through the equality of the asymptotic self-similarity indices:

$$1 - \frac{b}{\alpha} \;=:\; H_{\bar{X}} \;=\; H_Y \;:= 1 - \frac{1}{\beta} + \frac{1}{\alpha}$$

In particularly, the above correspondence between the parameters b and β is preserved in the decay rates of the codifference and the ruin probability.

- Limits of subordinated (nonlinear) functions?

- Invertibility?

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