Super-linear Processes

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Outline

Notation

- T is an MPT of $(\Omega, \mathcal{A}, P)$
- $\mathcal{F}_k$ is a filtratrion for which $\mathcal{F}_{k+1} = T^{-1}\mathcal{F}_k$
- J is a countable set
- $\forall j \in J,\ \xi_{i,j}, i \in \mathbb{Z}$, is a stationary sequence of martingale differences for which $E(\xi_{i,j}^2) = 1$.
- $\xi_{i,j} \perp \xi_{i',j'}$ when $j' \neq j$
- $c_{i,j},\ i \in \mathbb{Z},\ j \in J$, are square summable

Superlinear Processes

Then

$$X_k = \sum_{j \in J} \sum_{i \in \mathbb{Z}} c_{i,j} \xi_{k-i,j} = \sum_{j \in J} \left[\sum_{i \in \mathbb{Z}} c_{i,j} \xi_{k-i,j} \right]$$

is called a *super-linear process*. Then

- $\xi_{i,j},\ i \in \mathbb{Z},\ j \in J$, are *innovations*
- *Independent* innovations $\xi_{i,j} \sim F_j, i \in \mathbb{Z}, j \in J$
- The process is *causal* if $c_{i,j} = 0$ for $i < 0$
- The process is *linear* if J is a singleton

Herndorff's Example

There is a strongly mixing super linear process with independent innovations for which X_k are orthogonal and

$$\lim_{n\to\infty} P[X_1 + \cdots + X_n = 0] \geq \frac{1}{2}.$$

In the construction

- $\xi_{i,j}$ have large values for large j
- $c_{i,j}$ have a finite range for each j
- $\sum_{i\in\mathbb{Z}} c_{i,j} = 0$ for each j

Regular Processes

A stationary process is $\mathcal{F}_k$-*regular* if

- Each X_k is $\mathcal{F}_\infty$-measurable a
- $E(X_k|\mathcal{F}_{-\infty}) = E(X_k)$.

Proposition

Any mean 0 finite variance regular process is a super linear process.

From the Proof

Let

$$\mathcal{H}_i = L^2(\Omega, \mathcal{F}_i, P) \ominus L^2(\Omega, \mathcal{F}_{i-1}, P),$$

the orthogonal complement, and let

$$Q_i Y = E(Y|\mathcal{F}_i) - E(Y|\mathcal{F}_{i-1}),$$

the projection Y on $\mathcal{H}_i$ for $Y \in L^2(P)$. Then

$$X_k = \sum_{i \in \mathbb{Z}} Q_i X_k$$

Let e_j, $j \in J$, be an o.n. basis for $\mathcal{H}_0$, and $\xi_{i,j} = e_j \circ T^i, j \in J$.

Example

Let $\cdots \epsilon_{-1}, \epsilon_0, \epsilon_1, \cdots$ be i.i.d. 0 or 1 with probability $1/2$ each,

$$W_k = \sum_{i=0}^{\infty} (\frac{1}{2})^{i+1} \epsilon_{k-i},$$

and $X_k = g(W_k)$, where $g \in L_0^2(\lambda)$. If

$$g(w) = \sum_{r \in \mathbb{Z}} a_r e^{2\pi \imath r w},$$

and $\mathcal{F}_k = \sigma\{\cdots \epsilon_{k-1}, \epsilon_k\}$, then

$$Q_0 g(w) = \sum_{r \in \mathrm{Odd}} a_r e^{2\pi \imath r w}.$$

Sums

Write

$$X_k = \sum_{j \in J} \sum_{i \in \mathbb{Z}} c_{k-i,j} \xi_{i,j},$$
$$S_n = X_1 + \cdots + X_n.$$

Then

$$S_n = \sum_{j \in J} \sum_{i \in \mathbb{Z}} [b_{n-i,j} - b_{-i,j}] \xi_{i,j}$$

where $b_{n,j} = -(c_{n+1,j} + \cdots + c_{0,j}),\ 0$, or $c_{1,j} + \cdots + c_{n,j}$ for $n < 0,\ = 0$, or > 0.

Sums

So,

$$\sigma_n^2 = E(S_n^2) = \sum_{j \in J} \sum_{i \in \mathbb{Z}} [b_{n-i,j} - b_{-i,j}]^2.$$

Suppose

$$\lim_{n \to \infty} \sigma_n^2 = \infty.$$

Let

$$\bar{b}_{n,j} = \frac{b_{-n,j} + \cdots + b_{n,j}}{n}.$$

Then

$$\bar{\mathbf{b}}_n = (b_{n,j} : j \in J) \in \ell^2(J).$$

A Central Limit Theorem

Theorem 1

Suppose that $\sigma_n^2 \to \infty$. If

$$\sum_{j \in J} \left[\sum_{i=1}^{\infty} [b_{n-i,j} - b_{-i,j}]^2 + \sum_{i=1}^{\infty} [b_{n+i,j} - b_{i,j}]^2 \right] = o(\sigma_n^2) \qquad (*)$$

and the sequence

$$\mathbf{u}_n = \frac{\bar{\mathbf{b}}_n}{\|\bar{\mathbf{b}}_n\|}, \; n \geq 1,$$

is precompact in $\ell^2(J)$, then

A Central Theorem: Continued

Theorem 1: Continued

$$\frac{S_n}{\sigma_n} \Rightarrow Z \sim \Phi. \qquad (\dagger)$$

Conversely, if (*) and (†) for all F_j in the independent case, then $\bar{\mathbf{u}}_n, \ n \geq 1$, is precompact. (Recall: $\xi_{i,j} \sim F_j$)

Notes

- Conditions only restrict the coefficients $c_{i,j}$
- Best possible among such conditions

From the Proof

Let

$$D_{n,k} = \sum_{j \in J} \bar{b}_{n,j} \xi_{k,j},$$
$$M_{n,k} = D_{n,1} + \cdots + D_{n,k},$$

If $\sigma_n \to \infty$ and (*) holds, then

$$\max_{k \le n} \|S_k - M_{n,k}\|_2 = o(\sigma_n),$$

and, therefore,

$$\frac{S_n}{\sigma_n} \Rightarrow \Phi \quad \text{iff} \quad \frac{M_{n,n}}{\sigma_n} \Rightarrow \Phi.$$

From the Proof

So, $S_n/\sigma_n \Rightarrow \Phi$ if $D_{n,k}$ satisfy the conditions of the Martingale CLT:

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{k=1}^{n} E(D_{nk}^2 | \mathcal{F}_{k-1}) = 1 \tag{stbl}$$

and

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{k=1}^{n} E(D_{nk}^2 \mathbf{1}_{|D_{n,k}| \geq \epsilon\sigma_n} | \mathcal{F}_{k-1}) = 0. \tag{lf}$$

From (*), $\sigma_n^2 \sim \|\bar{\mathbf{b}}_n\|^2 \times n$ and, therefore,

From the Proof

$$\frac{M_{n,n}}{\sigma_n} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{D_{n,k}}{\|\bar{\mathbf{b}}_n\|} + o_p(\sigma_n)$$

and

$$\frac{D_{n,k}}{\|\bar{\mathbf{b}}_n\|} = \sum_{j \in J} u_{n,j} \xi_{k,j}$$

where $\mathbf{u}_n = \bar{\mathbf{b}}_n / \|\bar{\mathbf{b}}_n\|$.

From the Proof

If

$$\mathbf{u} = \lim_{n\to\infty} \mathbf{u}_n \text{ in } \ell^2(J),$$

then

$$\lim_{n\to\infty} D_{n,k} = \sum_{j\in J} u_j \xi_{k,j} \text{ in } L^2(P),$$

and the conditions of the Martingale CLT are easily checked.

For the relatively compact case, consider subsequences.

A NASC for the CLT

Theorem 2

Now consider a superlinear process with independent innovations. If $\sigma_n^2 \to \infty$ and

$$\sum_{j \in J} \left[\sum_{i=1}^{\infty} [b_{n-i,j} - b_{-i,j}]^2 + \sum_{i=1}^{\infty} [b_{n+i,j} - b_{i,j}]^2 \right] = o(\sigma_n^2), \qquad (*)$$

then $S_n/\sigma_n \Rightarrow \Phi$ iff

$$L_n^*(\epsilon) = \frac{1}{\|\bar{\mathbf{b}}_n\|^2} \sum_{j \in J} \bar{b}_{n,j}^2 \int_{|\bar{b}_{n,j} z| > \epsilon \ \|\bar{\mathbf{b}}_n\| \sqrt{n}} z^2 F_j\{dz\} \to 0$$

for each $\epsilon > 0$.

Examples

Suppose that $\sigma_n^2 \sim \|\bar{\mathbf{b}}_{n,j}\|^2 \times n \to \infty$.

1. If $\xi_{0,j},\ j \in J$, are uniformly integrable, then we need

$$\lim_{n\to\infty} \max_{j\in J} \frac{|\bar{\mathbf{b}}_{n,j}|}{\sigma_n} = 0.$$

2. If $\xi_{0,j} = \pm 2^{\frac{1}{2}j}$ with probability 2^{-j-1} each and $\xi_{0,j} = 0$ otherwise, then we need

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{|\bar{\mathbf{b}}_{n,j}|>\epsilon 2^{-\frac{1}{2}j}\sigma_n} |\bar{\mathbf{b}}_{n,j}|^2 = 0.$$

From the Proof

Observe:

$$\frac{S_n}{\sigma_n} = \sum_{j\in J}\sum_{i\in\mathbb{Z}}\left(\frac{b_{n-i,j} - b_{-i,j}}{\sigma_n}\right)\xi_{i,j} = \sum_{j\in J}\sum_{i\in\mathbb{Z}} Y_{n,i,j},$$

say. The Lindeberg Feller Condition is

$$L_n(\epsilon) := \sum_{j\in J}\sum_{i\in\mathbb{Z}}\int_{|Y_{n,i,j}|\geq\epsilon} Y_{n,i,j}^2 dP \to 0$$

and $n \to \infty$. It is necessary to relate $L_n(\epsilon)$ to

$$L_n^*(\epsilon) = \frac{1}{\|\bar{\mathbf{b}}_n\|^2}\sum_{j\in J}\bar{b}_{n,j}^2\int_{|\bar{b}_{n,j}z|>\epsilon\ \|\bar{\mathbf{b}}_n\|\sqrt{n}} z^2 F_j\{dz\}.$$

From the Proof

That is, it is necessary to related the distribution of $D_{n,k} = \sum_{j\in J} b_{n,j}\xi_{k,j}$ to the F_j. Apply the following to $Z_j = b_{n,j}\xi_{k,j}$.

A Baum Katz Inequality

Let $Z_j, j \in J$, *be independent random variables with means* 0 *and variances* $b_j^2, j \in J$, *for which* $\sum_{j\in J} b_j^2 < \infty$; *and let* $Y = \sum_{j\in J} Z_j$ *and* $\mathbf{b} = (b_j : j \in J)$. *Then, for all* $x > 0$,

$$P\left[|Y| > 3x\right] \le \left(\frac{\|\mathbf{b}\|^2}{x^2}\right)^2 + \sum_{j\in J} P\left[|Z_j| > x\right]$$

Conditional Normality

The CCLT

Suppose that $\sigma_n^2 := E(S_n^2) \to \infty$ and let

$$\Phi_n(\omega; z) = P\left[\frac{S_n}{\sigma_n} \leq z | \mathcal{F}_0\right](\omega).$$

Then the conditional central limit theorem holds if $\Phi_n \Rightarrow^p \Phi$ (in the Levy Metric say).

Conditional Normality:Continued

Known Result

For a causal linear process ($J = \{0\}$), the CCLT holds iff

$\|E(S_n|\cdots\xi_{-1},\xi_0)\|_2 = o(\sigma_n)$, or equivalently

$$\sum_{i=0}^{\infty}[b_{i+n} - b_i]^2 = o[\sum_{i=1}^{n} b_i^2].$$

Note: Here $\xi_i = \xi_{i,0}$ and $b_n = b_{n,0}$.

A CCLT

Theorem 3

For a causal superlinear process with independent innovations:

If $\sigma_n^2 \to \infty$ then the CCLT holds iff

$$\sum_{i=0}^{\infty} \|\mathbf{b}_{i+n} - \mathbf{b}_i\|_2^2 = o\left[\sum_{i=1}^{n} \|\mathbf{b}_i\|_2^2\right]$$

and

$$L_n^*(\epsilon) = \frac{1}{\|\bar{\mathbf{b}}_n\|^2} \sum_{j \in J} \bar{b}_{n,j}^2 \int_{|\bar{b}_{n,j} z| > \epsilon \ \|\bar{\mathbf{b}}_n\| \sqrt{n}} z^2 F_j\{dz\} \to 0$$

for each $\epsilon > 0$.

About the Proof

Combines

- Theorem 2
- NASC of Wu and W. (2004, Ann. Prob.)

THE END