Between the LIL and the LSL for random fields

Allan Gut

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Paris, June 22, 2010

Brașov

Brașov

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Who is M. Peligrad?

Looking around ....

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Who is M. Peligrad?

Looking around ....

.... among all men ....

Who is M. Peligrad?

Looking around ....

.... among all men ....

.... I finally find

Magda

Martingales and Amarts

ACCADEMIA NAZIONALE DEI LINCEI

Estratto dai *Rendiconti della Classe di Scienze fisiche, matematiche e naturali*

Serie VIII, vol. LVII, fasc. 1-2 (Luglio-Agosto) - Ferie 1974

Probabilità. — *Properties of uniform integrability and convergence for families of random variables.* Nota (*) di MAGDA RUBINSTEIN, presentata dal Socio B. SEGRE.

RIASSUNTO. — Sotto opportune condizioni, vien stabilita l'uniforme integrabilità di una famiglia di variabili casuali. Si generalizza inoltre un ben noto risultato sulle sottomartingale.

1. INTRODUCTION

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_n)_{n \in N}$ an increasing family of sub σ-fields of $\mathscr{F}$. In what follows $(X_n)_{n \in N}$ is a sequence of random variables such that:

$$X_n \quad \text{is} \quad \mathscr{F}_n\text{–measurable,} \qquad \text{and}$$

$$(1) \qquad \sum_{K=1}^{\infty} \int |E(Y_{K+1} \mid \mathscr{F}_K)| < \infty \qquad \text{where} \quad Y_{K+1} = X_{K+1} - X_K .$$

Mamaliga brinza shkvarki

Sarmale și mamaliga

Braşov aug 82

Bernoulli Conference, Chapel Hill 1994

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Sweden 1995

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Journal of Theoretical Probability, Vol. 12, No. 1, 1999

Almost-Sure Results for a Class of Dependent Random Variables[1]

Magda Peligrad[2,4] and Allan Gut[3]

Received January 30, 1997; revised August 7, 1998

The aim of this note is to establish almost-sure Marcinkiewicz-Zygmund type results for a class of random variables indexed by $\mathbb{Z}_+^d$—the positive d-dimensional lattice points—and having maximal coefficient of correlation strictly smaller than 1. The class of applications include filters of certain Gaussian sequences and Markov processes.

KEY WORDS: Random field; moment inequality; strong law; identically distributed random variables; maximal coefficient of correlation.

And now something different

Please remember that all of the following

is joint work with

Ulrich Stadtmüller, Ulm University

The LIL-LSL part is also joint with

Fredrik Jonsson, Uppsala University

Background

$X, X_1, X_2, \dots$ i.i.d. $\quad S_n = \sum_{k=1}^{n} X_k,\ n \geq 1.$

Windows — delayed sums

$$T_{n,n+k} = \sum_{j=n+1}^{n+k} X_j, \quad k \geq 1.$$

LLN Chow (1973)

$$\lim_{n\to\infty} \frac{T_{n,n+n^\alpha}}{n^\alpha} = 0 \quad \text{a.s.} \qquad \Longleftrightarrow \quad E|X|^{1/\alpha} < \infty,\ E\,X = 0$$

Background

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L**S**L — Lai (1974)

$$\limsup_{n\to\infty} \frac{T_{n,n+n^\alpha}}{\sqrt{2n^\alpha \log n}} = \sigma\sqrt{1-\alpha} \quad \text{a.s.}$$

$$\Longleftrightarrow$$

$$E\left(|X|^{2/\alpha}\left(\log^+|X|\right)^{-1/\alpha}\right) < \infty, \; EX^2 = \sigma^2, \; EX = 0.$$

Law of the **single** logarithm

Multiindex

$\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ i.i.d. $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbb{Z}_+^d$.

Partial order $\leq$ coordinate-wise.

$\mathbf{n}^\alpha$ coordinate-wise α-powers.

$\mathbf{n} \to \infty$ means $n_i \to \infty$ all i, $|\mathbf{n}| = \prod_{i=1}^d n_i$.

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Tail probabilities and moments

$$d(j) = \mathrm{Card}\,\{\mathbf{k} : |\mathbf{k}| = j\} = o(j^\delta), \ \forall\, \delta > 0,$$

$$M(j) = \mathrm{Card}\,\{\mathbf{k} : |\mathbf{k}| \le j\} \to \frac{j(\log j)^{d-1}}{(d-1)!}.$$

Partial summation $\Longrightarrow$

$$\sum_{\mathbf{n}} P(|X| > |\mathbf{n}|) \sim E\,M(|X|) \sim E\,|X|(\log^+ |X|)^{d-1}.$$

The law of the iterated logartihm — (LIL)

$$\limsup_{\mathbf{n}\to\infty} \frac{S_{\mathbf{n}}}{\sqrt{2|\mathbf{n}|\log\log|\mathbf{n}|}} = \sigma\sqrt{d} \quad \text{a.s.}$$

$$\Longleftrightarrow$$

$$\begin{cases} E\,X = 0, \quad E\,X^2 = \sigma^2, & \text{when} \quad d = 1, \\ E\,X^2 \dfrac{(\log^+|X|)^{d-1}}{\log^+\log^+|X|} < \infty, \quad \text{and} & \\ E\,X = 0, \quad E\,X^2 = \sigma^2, & \text{when} \quad d \geq 2. \end{cases}$$

$d = 1$: Hartman-Wintner (1941) (sufficiency)

Strassen (1966) (necessity)

$d \geq 2$: Wichura (1973)

Remark

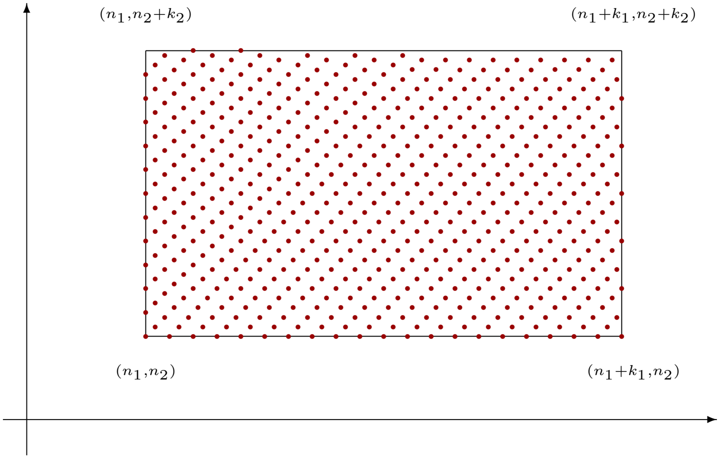
Converse to LIL easy when $d \geq 2$, since

$$
\begin{aligned}
& \text{LIL} \\
\Longrightarrow \quad & \frac{X_{\mathbf{n}}}{\sqrt{|\mathbf{n}| \log\log|\mathbf{n}|}} \xrightarrow{a.s.} 0 \\
\Longleftrightarrow \quad & \sum_{\mathbf{n}} P(|X| > \sqrt{|\mathbf{n}| \log\log|\mathbf{n}|}) < \infty \\
\Longleftrightarrow \quad & \sum_{j} d(j) P(|X| > \sqrt{j \log\log j}) < \infty \\
\Longleftrightarrow \quad & E\,X^2 \frac{(\log^+|X|)^{d-1}}{\log^+\log^+|X|} < \infty.
\end{aligned}
$$

Not enough when $d = 1$!

A typical window for $d = 2$

$$T_{\mathbf{n},\mathbf{n}+\mathbf{k}} = S_{n_1+k_1,n_2+k_2} - S_{n_1+k_1,n_2} - S_{n_1,n_2+k_2} + S_{n_1,n_2}$$



Theorem

$\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ i.i.d., $\quad E\,X = 0,\ \mathrm{Var}\,X = \sigma^2,$

$S_{\mathbf{n}} = \sum_{\mathbf{k} \le \mathbf{n}} X_{\mathbf{k}},\ \mathbf{n} \in \mathbb{Z}_+^d, \qquad 0 < \alpha < 1.$

If

$$E\,X^{2/\alpha}(\log^+|X|)^{d-1-1/\alpha} < \infty \tag{1}$$

then

$$\limsup_{\mathbf{n}\to\infty} \frac{T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}}{\sqrt{2|\mathbf{n}|^\alpha \log|\mathbf{n}|}} = \sigma\sqrt{1-\alpha} \text{ a.s.} \tag{2}$$

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Conversely, if

$$P\Big(\limsup_{\mathbf{n}\to\infty} \frac{|T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}|}{\sqrt{|\mathbf{n}|^\alpha \log|\mathbf{n}|}} < \infty\Big) > 0, \tag{3}$$

then (1) holds, $E\,X = 0$, and (2) holds with $\sigma^2 = \mathrm{Var}\,X$. □

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Sketch of proof

δ small,

$$b_{\mathbf{n}} = b_{|\mathbf{n}|} = \frac{\sigma\delta}{\varepsilon}\frac{\sqrt{|\mathbf{n}|^{\alpha}}}{\log|\mathbf{n}|},$$

$$\begin{aligned}
X'_{\mathbf{n}} &= X_{\mathbf{n}} I\{|X_{\mathbf{n}}| \le b_{\mathbf{n}}\},\\
X''_{\mathbf{n}} &= X_{\mathbf{n}} I\{b_{\mathbf{n}} < |X_{\mathbf{n}}| < \delta\sqrt{|\mathbf{n}|^{\alpha}\log|\mathbf{n}|}\},\\
X'''_{\mathbf{n}} &= X_{\mathbf{n}} I\{|X_{\mathbf{n}}| \ge \delta\sqrt{|\mathbf{n}|^{\alpha}\log|\mathbf{n}|}\}.
\end{aligned}$$

Sketch of proof

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- $S'_{\mathbf{n}}$, $S''_{\mathbf{n}}$, $S'''_{\mathbf{n}}$, etc ;
- $E\,X'_{\mathbf{n}}$, $E\,X''_{\mathbf{n}}$, $E\,X'''_{\mathbf{n}}$ "small" ;
- $E\,S'_{\mathbf{n}}$, $E\,S''_{\mathbf{n}}$, $(E\,S'''_{\mathbf{n}})$ "small" ;
- $\mathrm{Var}\,(T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}) \approx \mathbf{n}^\alpha\sigma^2$.

Exponential bounds; $T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}$

Upper bound

$$P(|T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}| > \varepsilon\sqrt{2|\mathbf{n}|^\alpha \log|\mathbf{n}|})$$

$$\leq P(|T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha} - ET'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}| > \varepsilon(1-\delta)\sqrt{2|\mathbf{n}|^\alpha \log|\mathbf{n}|})$$

$$\leq 2|\mathbf{n}|^{-\frac{\varepsilon^2}{\sigma^2}\cdot(1-\delta)^3}, \qquad |\mathbf{n}| \text{ large.}$$

Exponential bounds; $T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}$

Upper bound

$$P(|T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}| > \varepsilon\sqrt{2|\mathbf{n}|^\alpha \log|\mathbf{n}|})$$

$$\leq P(|T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha} - ET'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}| > \varepsilon(1-\delta)\sqrt{2|\mathbf{n}|^\alpha \log|\mathbf{n}|})$$

$$\leq 2|\mathbf{n}|^{-\frac{\varepsilon^2}{\sigma^2}\cdot(1-\delta)^3}, \qquad |\mathbf{n}| \text{ large.}$$

Lower bound

$$P(T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha} > \varepsilon\sqrt{2|\mathbf{n}|^\alpha \log|\mathbf{n}|})$$

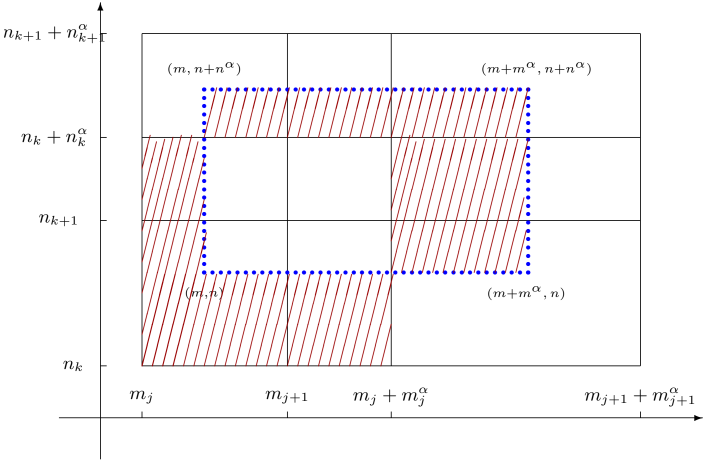
$$\geq P(T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha} - ET'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha} > \varepsilon(1+\delta)\sqrt{2|\mathbf{n}|^\alpha \log|\mathbf{n}|})$$

$$\geq |\mathbf{n}|^{-\frac{\varepsilon^2}{\sigma^2}\cdot\frac{(1+\delta)^2(1+\gamma)}{(1-\delta)}}, \qquad |\mathbf{n}| \text{ large}, \quad \gamma \text{ small} > 0.$$

Same procedure as every year

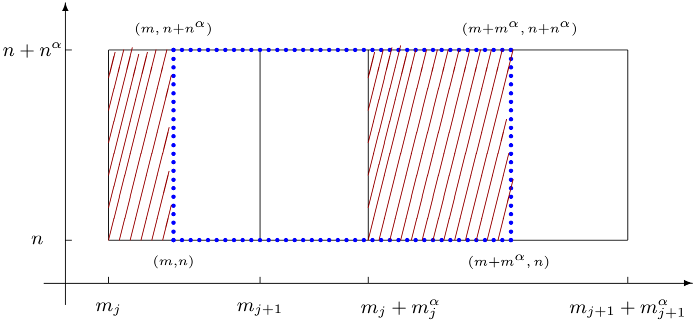
1. Dispose of $T''_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}$;
2. Dispose of $T'''_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}$;
3. Upper exponential for subsequence of $T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}$;
4. Borel-Cantelli 1 $T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}$ OK;
5. Filling gaps;
6. $1+2+4+5 \quad \Longrightarrow \quad \limsup T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha} \leq \cdots$;
7. Lower exponential for subsequence of $T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}$;
8. $\rightarrow$ increments;
9. Borel-Cantelli 2 $T'_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}$ OK;
10. $1+2+9 \quad \Longrightarrow \quad \limsup T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha} \geq \cdots$;
11. $6+10 \quad \Longrightarrow \quad \limsup T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha} = \cdots$;
12. $\square$

Upper bound; Filling gaps — center



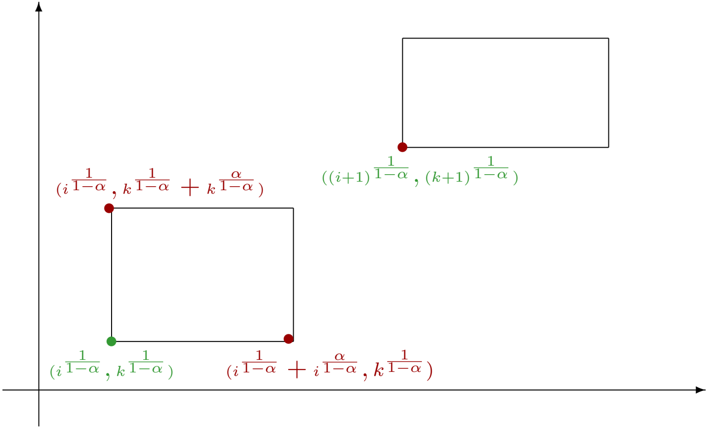
Upper bound; Filling gaps — boundary

We first need a denser subsequence, and then:



Lower bound

Independence of windows (B-C 2) $\Longleftrightarrow$ Disjointness



Finally!

$$\limsup_{\mathbf{n}\to\infty} \frac{|T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}|}{\sqrt{2|\mathbf{n}|^\alpha \log|\mathbf{n}|}} = \sigma\sqrt{1-\alpha} \quad \text{a.s.}$$

which proves the sufficiency.

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Necessity

LSL $\Longrightarrow$

$$\limsup_{n\to\infty} \frac{|X_{\mathbf{n}}|}{\sqrt{|\mathbf{n}|^{\alpha}\log|\mathbf{n}|}} < \infty \quad \text{a.s.,}$$

so that Borel-Cantelli 2 $\Longrightarrow$

$$\begin{aligned} \infty &> \sum_{\mathbf{n}} P(|X_{\mathbf{n}}| > \sqrt{|\mathbf{n}|^{\alpha}\log|\mathbf{n}|}) \\ &= \sum_{\mathbf{n}} P(|X| > \sqrt{|\mathbf{n}|^{\alpha}\log|\mathbf{n}|}), \end{aligned}$$

$\Longleftrightarrow$

$$E\,X^{2/\alpha}(\log^{+}|X|)^{d-1-1/\alpha} < \infty.$$

Finished!

Fredrik: What happens if the α's are different?

Now:

$$\alpha \longrightarrow \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d)$$

$$\mathbf{n}^{\alpha} \longrightarrow \mathbf{n}^{\boldsymbol{\alpha}} = (n_1^{\alpha_1}, n_2^{\alpha_2}, \ldots, n_d^{\alpha_d})$$

$$|\mathbf{n}|^{\alpha} \longrightarrow |\mathbf{n}^{\boldsymbol{\alpha}}| = \prod_{k=1}^{d} n_k^{\alpha_k}$$

Theorem

$\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ i.i.d., $\quad E\,X = 0,\ \operatorname{Var} X = \sigma^2.$

$0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d < 1 \quad$ and $\quad p = \max\{k : \alpha_k = \alpha_1\}.$

If

$$E|X|^{2/\alpha_1}(\log^+|X|)^{p-1-1/\alpha_1} < \infty,$$

then

$$\limsup_{\mathbf{n}\to\infty} \frac{T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}}{\sqrt{2|\mathbf{n}|^\alpha \log|\mathbf{n}|}} = \sigma\sqrt{1-\alpha_1} \quad \text{a.s.}$$

And "conversely". □

What about the degenerate case; $\alpha = 0$?

$$0 = \alpha_1 \le \alpha_2 \le \cdots \le \alpha_d < 1$$

$$q = \max\{k : \alpha_k = 0\} \qquad r = \max\{k : \alpha_k = \alpha_{q+1}\}.$$

If

$$E|X|^{2/\alpha_{q+1}} (\log^+ |X|)^{r-q-1-1/\alpha_{q+1}} < \infty,$$

then

$$\limsup_{\mathbf{n}\to\infty} \frac{T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}}{\sqrt{2|\mathbf{n}|^\alpha \log|\mathbf{n}|}} = \sigma\sqrt{1-\alpha_{q+1}} \quad \text{a.s.}$$

And conversely.

In particular $\quad q = 0, r = d \quad$ and $\quad q = 0, r = p.$

What about the boundary case; $\alpha = 1$?

Now:

$$n^{\alpha} \quad \longrightarrow \quad n/\log n, \quad n/\log\log n, \quad \ldots$$

More generally

$$n^{\alpha} \quad \longrightarrow \quad a_n = n/L(n),$$

where $L \in \mathcal{SV}$ is differentiable and $\cdots\cdots$.

What about the boundary case; $\alpha = 1$?

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$$n^\alpha \quad \longrightarrow \quad n/\log n, \quad n/\log\log n, \quad \dots$$

More generally

$$n^\alpha \quad \longrightarrow \quad a_n = n/L(n),$$

where $L \in \mathcal{SV}$ is differentiable and $\cdots\cdots$.

Moreover,

$$f(n) = \min\{a_n \cdot d_n,\, n\},$$

where

$$d_n = \log \frac{n}{a_n} + \log\log n = \log L(n) + \log\log n.$$

Between LIL and LSL

Theorem

$\{X_k, k \geq 1\}$ i.i.d., $E\,X = 0$, $\operatorname{Var} X = \sigma^2$.

If

$$E\,f^{-1}(X^2) < \infty,$$

then

$$\limsup_{n\to\infty} \frac{T_{n,n+a_n}}{\sqrt{2a_n d_n}} = \sigma \quad \text{a.s.}$$

And conversely.

Two "immediate" examples

$L(n) = \log n$

$$\limsup_{n\to\infty} \frac{T_{n,n+n/\log n}}{\sqrt{4\frac{n}{\log n}\log\log n}} = \sigma \quad \text{a.s.}$$

$$\iff \qquad E\,X^2 \frac{\log^+ |X|}{\log^+ \log^+ |X|} < \infty.$$

Two "immediate" examples

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$$\limsup_{n\to\infty} \frac{T_{n,n+n/\log n}}{\sqrt{4\frac{n}{\log n}\log\log n}} = \sigma \quad \text{a.s.}$$

$$\iff \qquad E\,X^2 \frac{\log^+ |X|}{\log^+ \log^+ |X|} < \infty.$$

$L(n) = \log_m n, \quad m = 2, 3, \ldots$

$$\limsup_{n\to\infty} \frac{T_{n,n+n/\log_m(n)}}{\sqrt{2\frac{n}{\log_m(n)}\log\log n}} = \sigma \quad \text{a.s.}$$

$$\iff \qquad E\,X^2 = \sigma^2 < \infty.$$

Remarks on the proof

- LIL-type truncations.
- **Lemma** (Fredrik)

  Set $\varphi(y) = \int^y \frac{L(u)\,du}{u}$. Then

  $$\frac{\log(L(t)\log t)}{\log \varphi(t)} \to 1 \quad \text{as} \quad t \to \infty,$$

  so that $d_{n_k} \sim \log k$ as $k \to \infty$.
- LIL-type proof:

  T' via exp. bounds

  T'' the "thin piece" via complicated analysis

  T''' via moment assumption.
- Then the entire sequence.

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Two further examples

$$L(n) = (\log n)^p / (\log\log n)^q, \quad p, q > 0$$

$$\limsup_{n\to\infty} \frac{T_{n,n+n(\log\log n)^q/(\log n)^p}}{\sqrt{2(p+1)\frac{n}{(\log n)^p}(\log\log n)^{q+1}}} = \sigma \quad \text{a.s.}$$

$$\iff \qquad E\,X^2 \frac{(\log^+|X|)^p}{(\log^+\log^+|X|)^{q+1}} < \infty.$$

Two further examples

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$$\Longleftrightarrow \qquad E\,X^2 \frac{(\log^+|X|)^p}{(\log^+\log^+|X|)^{q+1}} < \infty.$$

$$L(n) = \exp\{\sqrt{\log n}\}$$

$$\limsup_{n\to\infty} \frac{T_{n,n+n/\exp\{\sqrt{\log n}\}}}{\sqrt{2\frac{n}{\exp\{\sqrt{\log n}\}}\sqrt{\log n}}} = \sigma \quad \text{a.s.}$$

$$\Longleftrightarrow \qquad E\,X^2 \frac{\exp\{\sqrt{2\log^+|X|}\}}{\sqrt{\log^+|X|}} < \infty.$$

Additional variations

- A multivariate version of the "between case";
- Different growth rates of the L-functions;
- Mixtures of LSL and "between".
- Etc ?

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- A multivariate version of the "between case";
- Different growth rates of the L-functions;
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- Etc ?

Some examples:

- $T_{m+m/\log m,\, n+n/\log n}$;
- $T_{m+m/\log\log m,\, n+n/\log\log n}$;
- $T_{m+m/\log m,\, n+n/\log\log n}$;
- $T_{m+m^\alpha,\, n+n/\log n}$;
- Etc ?

Results — same growth rate

$\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^2\}$ i.i.d.

$$\limsup_{m,n\to\infty} \frac{T_{(m,n),(m+m/\log m, n+n/\log n)}}{\sqrt{4mn\frac{\log\log m+\log\log n}{\log m \log n}}} = \sigma \qquad \text{a.s.}$$

$$\Longleftrightarrow \qquad E\,X^2 \frac{(\log^+|X|)^3}{\log^+\log^+|X|} < \infty,$$

Results — same growth rate

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$$\iff \quad E\,X^2 \frac{(\log^+|X|)^3}{\log^+\log^+|X|} < \infty,$$

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$$\iff \quad E\,X^2 \log^+|X| \log^+\log^+|X| < \infty,$$

and, in both cases, $E\,X = 0$, $E\,X^2 = \sigma^2$.

Results — different growth rates

$$\limsup_{m,n\to\infty} \frac{T_{(m,n),(m+m/\log m,\, n+n/\log\log n)}}{\sqrt{4mn\frac{\log\log m+\log\log n}{\log m \log\log n}}} = \sigma \quad \text{a.s.}$$

$$\iff \quad E\,X^2(\log^+|X|)^2 < \infty,$$

Results — different growth rates

$$\limsup_{m,n\to\infty} \frac{T_{(m,n),(m+m/\log m,\, n+n/\log\log n)}}{\sqrt{4mn\frac{\log\log m+\log\log n}{\log m \log\log n}}} = \sigma \quad \text{a.s.}$$

$$\iff \quad E\,X^2(\log^+|X|)^2 < \infty,$$

and, for $0 < \alpha < 1$,

$$\limsup_{m,n\to\infty} \frac{T_{(m,n),(m+m^\alpha,\, n+n/\log n)}}{\sqrt{2m^\alpha n\frac{(1-\alpha)\log(mn)}{\log n}}} = \sigma \quad \text{a.s.}$$

$$\iff \quad E\,X^{2/\alpha}(\log^+|X|)^{-1/\alpha} < \infty,$$

and, in both cases, $E\,X = 0$, $E\,X^2 = \sigma^2$.

A Question to Magda

what about dependent cases?

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what about dependent cases?

For example,

interlaced rho-mixing?

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Dear Magda

Best wishes !

Allt gott !

Meilleurs voeux !

Alles Gute !

Mazal tov !